Lecture slide No.3 for Math 207c
What happens if $k = 1$?

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A Hecke eigenform f of weight $k = 1$ is associated to a 2-dimensional Artin representation $\rho = \rho_f : G \to \mathrm{GL}_2(\mathbb{Z}[f])$ with finite image (by Deligne–Serre). Thus the system $\{\rho_{f,\mathfrak{p}} = \rho_f\}_{\mathfrak{p}}$ is a singleton, but still we can vary $\mathfrak{p}$ and regard ρ_f as having values in $\mathrm{GL}_2(W_{\mathfrak{p}})$.

The situation is totally different from the case of $k \geq 2$. We explore dependence on $\mathfrak{p}$ of the universal rings $R_{\mathfrak{p}}^{ord}$ and $R_{\chi,\mathfrak{p}}$ representing the deformation functors $\mathcal{D} = \mathcal{D}_{\mathfrak{p}}, \mathcal{D}_\chi = \mathcal{D}_{\chi,\mathfrak{p}} : \mathcal{C} \to SETS$ of $\overline{\rho}_{\mathfrak{p}}$ defined in §0.22 (if $k = 1$, $\overline{\rho}_{\mathfrak{p}}$ satisfies (ord$_p$) for almost all $\mathfrak{p}$ as $|\rho(D_p)|$ is bounded by $\rho(G)$ independent of p). Write C for the Artin conductor of ρ_f.

§3.1. Representability of $\mathcal{D}_\chi$ by a Hecke algebra. If $k = 1$, we have $\chi = \psi$. We already mentioned, the following identity

$$R_{\mathfrak{p}}^{ord}/(t - \psi(\gamma)) \cong R_{\psi,\mathfrak{p}}.$$

We can define $\mathbb{T}_{\psi,\mathfrak{p}} := \mathbb{T}_{\mathfrak{p}}/(t - \psi(\gamma))\mathbb{T}_{\mathfrak{p}}$. Under the Taylor–Wiles condition: $\overline{\rho} \not\cong \mathrm{Ind}_{\mathbb{Q}[\sqrt{p^*}]}^{\mathbb{Q}} \overline{\varphi}$ (for whichever choice of $\overline{\varphi}$), we have $R_{\mathfrak{p}} \cong \mathbb{T}_{\mathfrak{p}}$ and hence $R_{\psi,\mathfrak{p}} \cong \mathbb{T}_{\psi,\mathfrak{p}}$. However $\mathbb{T}_1 \subset h_1(Cp, \psi)_{/W}$ is far smaller than $\mathbb{T}_{\psi,\mathfrak{p}}$, and therefore, $R_{\psi,\mathfrak{p}}$ does not have a canonical $\mathbb{Z}[\psi]$-integral structure. The reason for this is that the existence of the Eichler–Shimura isomorphism is only for $k \geq 2$:

$$H^1_!(\Gamma_0(Cp), \mathrm{S}ym^{k-2} \otimes \psi) \cong S_k(Cp, \psi)^2$$

as Hecke modules which is used to prove the rank theorem in §2.16 which is also only valid for $k \geq 2$. Here $\mathrm{S}ym^n$ is the symmetric n-th tensor representation of $\Gamma_0(Cp) \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ and we regard ψ as character of $\Gamma_0(Cp)$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \psi(d)$ naturally.

§3.2. Some remarks.

Since ρ_f is the deformation of $\overline{\rho}_{f,\mathfrak{p}}$, we have a natural surjection $\mathbb{T}_{\psi,\mathfrak{p}} \twoheadrightarrow \mathbb{T}_1$. It is the case to have rationality of L-values with respect to the period (which is independent of the choice of the place), we need to have three cohomological interpretation of modular/automorphic forms.

(1) Automorphic form as an analytic function;

(2) Automorphic form giving rise to topological (Betti) cohomology class of locally constant sheaf (Eichler–Shimura);

(3) Automorphic form giving rise to de Rham cohomology class (analytic cohomology).

All these requirements are fulfilled, an automorphic form is called algebraic (or cohomological). Weight 1 forms miss Betti property. It is also a reflection of non-criticality of $L(1, Ad(f))$ for f of weight 1. Artin L-value is non-critical if its splitting field has complex places.

§3.3. Selmer group over a number field.

For a subfield M of $F^{(p)}(\overline{\rho})$, writing G_M for the subgroup of G fixing M and $D_\wp \subset G_M$ for the decomposition subgroup of a prime $\wp|p$ of M, we define the Selmer group Sel$(Ad(\rho))$ for any $\rho \in \mathcal{D}(A)$ by

$$\mathrm{Sel}_M(Ad(\rho_A)) := \mathrm{Ker}(H^1(G_M, Ad(\rho_A)^*) \to \prod_{\wp|p} \frac{H^1(D_\wp, Ad(\rho_A)^*}{F^+_{-,\wp} H^1(D_\wp, Ad(\rho_A)^*)}),$$

where $\wp$ runs over all prime factors of p in M, and choosing $a_\wp \in \mathrm{GL}_2(A)$ so that $a_\wp \rho_A a_\wp^{-1}|_{D_\wp} = \begin{pmatrix} \epsilon_\wp & * \\ 0 & \delta_\wp \end{pmatrix}$ with $\delta_\wp$ unramified and $\delta_\wp$ mod $\mathfrak{m}_A = \overline{\delta}$, $a_\wp F^+_{-,\wp} H^1(D_\wp, Ad(\rho)^*) a_\wp^{-1}$ is made of cohomology classes *upper triangular over* $D_\wp$ *and upper nilpotent over the inertia subgroup* $I_\wp$ *of* $D_\wp$.

For a while, the weight k of f is an integer ≥ 1. Let $\overline{\rho} = \overline{\rho}_{\mathfrak{p}}$ and $\mathbb{T} = \mathbb{T}_{\mathfrak{p}}$.

§3.4. One generator theorem.

Theorem 3.4: *Suppose $\overline{\rho}|_{D_p} = \overline{\epsilon} \oplus \overline{\delta}$ (p tamely ramified). Assume that $Cl_M \otimes_{\mathbb{Z}[G]} Ad(\overline{\rho}) = 0$ for $M = F(Ad(\overline{\rho}))$ and that the Galois module $Ad(\overline{\rho})|_{D_p}$ does not contain $\mu_p(\overline{\mathbb{Q}}_p) \otimes_{\mathbb{F}_p} \mathbb{F}$ as a Galois subquotient. Then we have $\mathbb{T} \cong \Lambda[X]/(D(X))$ for a distinguished polynomial $D(X)$ with respect to $\mathfrak{m}_\Lambda$ (i.e., $\dim_{\mathbb{F}} t_{R^{ord}/\Lambda} = 1$).*

- $D(X)$ is distinguished if $D(X) \equiv X^{\deg(D)} \mod \mathfrak{m}_\Lambda$.
- If $k = 1$, $p \nmid |\mathrm{Im}(\rho_f)| \Rightarrow p$ is tamely ramified in $F(\overline{\rho})/\mathbb{Q}$.
- Suppose $p \nmid C$. For $2 \leq k \leq p$, p is tamely ramified if and only if there exists $g \in S_{p+1-k}(C, \psi)_{/\mathbb{F}}$ with $na(n, f) \equiv n^k a(n, g) \mod p$ for all n (Conjectured by Serre and a theorem of D. Gross and R. Coleman–J. F. Voloch; see Inventiones **110** (1992), 263–281). The form g is called a companion form of f. Weight 1 form is a companion of weight p form.
- Is the p-tameness necessary for the assertion of the theorem?

§**3.5. Induced representation.** We prepare several facts on induced representation for the proof of Theorem 3.4. Let $A \in CL_W$ and $\mathcal{G}$ be a profinite group with a closed subgroup $\mathcal{H}$. Put $\Delta := \mathcal{G}/\mathcal{H}$. Let $\mathcal{H}$ be a character $\varphi : \mathcal{G} \to A$. Let $A(\varphi) \cong A$ on which $\mathcal{H}$ acts by φ. Regard the group algebra $A[\mathcal{G}]$ as a left and right $A[\mathcal{G}]$-module by multiplication. Define $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \varphi_{/A} := A[\mathcal{G}] \otimes_{A[\mathcal{H}]} A(\varphi)$ (so, $\xi h \otimes a = \xi \otimes ha = \xi \otimes \varphi(h)a = \varphi(a)(\xi \otimes a))$ for $h \in \mathcal{H}$. and let $\mathcal{G}$ acts on $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \varphi_{/A}$ by $g(\xi \otimes a) := (g\xi) \otimes a$. The resulted $\mathcal{G}$-module $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \varphi_{/A}$ is the induced module.

Similarly we can think of $\mathrm{ind}_{\mathcal{H}}^{\mathcal{G}} \varphi_{/A} := \mathrm{Hom}_{A[\mathcal{H}]}(A[\mathcal{G}], A(\varphi))$ (so, $\phi(h\xi) = h\phi(\xi) = \varphi(h)\phi(\xi))$ on which $g \in \mathcal{G}$ acts by $g\phi(\xi) = \phi(\xi g)$. In some books, $\mathrm{ind}_{\mathcal{H}}^{\mathcal{G}} \varphi$ is written as $\mathrm{Coind}_{\mathcal{H}}^{\mathcal{G}} \varphi$ (co-induced representation), but they are isomorphic if $\mathcal{H}$ has finite index in $\mathcal{G}$ (as we will see soon).

§3.6. Matrix form of induced representations. Assuming $(\mathcal{G} : \mathcal{H}) = 2$ for simplicity, we like to describe matrix form of $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \varphi$. Suppose that φ has order prime to p. Then for $\varsigma \in \mathcal{G}$ generating $\mathcal{G}$ over $\mathcal{H}$, $\varphi_\varsigma(h) = \varphi(\varsigma^{-1} h \varsigma)$ is again a character of $\mathcal{H}$. The module $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \varphi$ has a basis $1_{\mathcal{G}} \otimes 1$ and $\varsigma \otimes 1$ for the identity element $1_{\mathcal{G}}$ of $\mathcal{G}$ and $1 \in A \cong A(\varphi)$.

We have

$$g(1_{\mathcal{G}} \otimes 1, \varsigma \otimes 1) = (g \otimes 1, g\varsigma \otimes 1)$$
$$= \begin{cases} (1_{\mathcal{G}} \otimes g, \varsigma \otimes \varsigma^{-1} g \varsigma) = (1_{\mathcal{G}} \otimes 1, \varsigma \otimes 1) \begin{pmatrix} \varphi(g) & 0 \\ 0 & \varphi_\varsigma(g) \end{pmatrix} & \text{if } g \in \mathcal{H}, \\ (\varsigma \otimes \varsigma^{-1} g, 1_{\mathcal{G}} \otimes g\varsigma) = (1_{\mathcal{G}} \otimes 1, \varsigma \otimes 1) \begin{pmatrix} 0 & \varphi(g\varsigma) \\ \varphi(\varsigma^{-1} g) & 0 \end{pmatrix} & \text{if } g\varsigma \in \mathcal{H}, \end{cases}$$

Thus extending φ to $\mathcal{G}$ by 0 outside $\mathcal{H}$, we get

$$\boxed{\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \varphi(g) = \begin{pmatrix} \varphi(g) & \varphi(g\varsigma) \\ \varphi(\varsigma^{-1} g) & \varphi(\varsigma^{-1} g \varsigma) \end{pmatrix}.} \quad (1)$$

§3.7. $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}\varphi \cong \mathrm{ind}_{\mathcal{H}}^{\mathcal{G}}\varphi$.

We prove now that the two inductions are equal: $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}\varphi \cong \mathrm{ind}_{\mathcal{H}}^{\mathcal{G}}\varphi$. The induction $\mathrm{ind}_{\mathcal{H}}^{\mathcal{G}}\varphi$ has basis (ϕ_1, ϕ_ς) over $A[\mathcal{H}]$ given by $\phi_1(\xi + \xi'\varsigma^{-1}) = \varphi(\xi) \in A = A(\varphi)$ and $\phi_\varsigma(\xi + \xi'\varsigma^{-1}) = \varphi(\xi') \in A = A(\varphi)$ for $\xi \in A[\mathcal{H}]$; so, $(*)$ $\phi_1(\xi' + \xi\varsigma^{-1}) = \phi_\varsigma(\xi + \xi'\varsigma^{-1})$. Then we have

$$
\begin{aligned}
&g(\phi_1(\xi + \xi'\varsigma^{-1}), \phi_\varsigma(\xi + \xi'\varsigma^{-1})) \\
&\qquad\qquad = (\phi_1(\xi g + \xi'\varsigma^{-1} g\varsigma\varsigma^{-1}), \phi_\varsigma(\xi g + \xi'\varsigma^{-1} g\varsigma\varsigma^{-1}))
\end{aligned}
$$

$$
= \begin{cases} (\phi_1(\xi), \varphi_\varsigma(\xi')) \begin{pmatrix} \varphi(g) & 0 \\ 0 & \varphi_\varsigma(g) \end{pmatrix} & (g \in \mathcal{H}), \\ (\phi_1(\xi'\varsigma^{-1} g), \phi_\varsigma(\xi g\varsigma)) \overset{(*)}{=} (\phi_1(\xi), \phi_\varsigma(\xi')) \begin{pmatrix} 0 & \varphi(g\varsigma) \\ \varphi(\varsigma^{-1} g) & 0 \end{pmatrix} & (g\varsigma \in \mathcal{H}). \end{cases}
$$

Thus we get

$$
\boxed{\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}\varphi \cong \mathrm{ind}_{\mathcal{H}}^{\mathcal{G}}\varphi.} \tag{2}
$$

§3.8. Adjoint formula for Hom**.**

Let T be an S-algebra (here T and S are possibly non-commutative rings with identity). Let M be an S-module and N be a T-module. Regard T as a right S-module by right multiplication, and consider the scalar extension $T \otimes_S M$ which is an T-module by $\alpha(a \otimes m) = (\alpha a) \otimes m$ for $\alpha, a \in T$. Let $i : M \to T \otimes_S M$ be the S-linear map $i(m) = 1_T \otimes m$. Since $i(bm) = 1 \otimes_S bm = b \otimes_S m = bi(m)$ for $b \in S$, indeed, i is S-linear. We have the following universal propertry

- If N is a T-module, for any S-linear map $M \xrightarrow{f} N$, there is a unique T-morphism $g : T \otimes_S M \to N$ such that $g \circ i = f$.

This follows from the universality of the tensor product applied to the T-bilinear map $T \otimes_S M \to N$ given by $a \otimes m \mapsto af(m)$. Therefore, we get the adjoint formula for the tensor product:

$$\mathrm{Hom}_T(T \otimes_S M, N) \cong \mathrm{Hom}_S(M, N).$$

§3.9. Dual and derived category version of adjoint.

By the derived category version of this, we get

$$\mathrm{Ext}^q_T(T \otimes_S M, N) \cong \mathrm{Ext}^q_S(M, N) \quad \text{for all } q \geq 0.$$

There is a dual version. Regard $\mathrm{Hom}_S(T, M)$ as T-module by $\alpha\phi(a) = \phi(a\alpha)$ for $\phi \in \mathrm{Hom}_S(T, M)$. Let $\pi : \mathrm{Hom}_S(T, M) \to M$ by $\pi(\phi) = \phi(1_T)$, which is S-linear. Then by the universality of Hom_S, we have

• If N is a T-module, for any S-linear map $N \xrightarrow{f} M$, there is a unique T-morphism $g : N \to \mathrm{Hom}_S(T, M)$ such that $\pi \circ g = f$.

From this, we get

$$\mathrm{Hom}_T(N, \mathrm{Hom}_S(T, M)) \cong \mathrm{Hom}_S(N, M),$$

and again

$$\mathrm{Ext}^q_T(N, \mathrm{Hom}_S(T, M)) \cong \mathrm{Ext}^q_S(N, M) \quad \text{for all } q \geq 0.$$

§**3.10. Shapiro's lemma.** Let $\mathcal{G}$ be a group and $\mathcal{H}$ be a subgroup of finite index. Take a commutative ring B with identity. We apply the above argument to the group algebras $T = B[\mathcal{G}]$ and $S = B[\mathcal{H}]$. Then we write $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} M := T \otimes_S M$ and $\mathrm{ind}_{\mathcal{H}}^{\mathcal{G}} M := \mathrm{Hom}_S(A, M)$ as T-modules. Then we have

Lemma 3.10.1 (A. Shapiro)

$$\mathrm{Ext}^q_{B[\mathcal{G}]}(\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} M, N) \cong \mathrm{Ext}^q_{B[\mathcal{H}]}(M, N) \quad \text{for all } q \geq 0,$$
$$\mathrm{Ext}^q_{B[\mathcal{G}]}(N, \mathrm{ind}_{\mathcal{H}}^{\mathcal{G}} M) \cong \mathrm{Ext}^q_{B[\mathcal{H}]}(N, M) \quad \text{for all } q \geq 0.$$

Since the cohomology group $H^q(\mathcal{G}, N)$ can be identified with $\mathrm{Ext}^n_{B[\mathcal{G}]}(B, N)$ (see [MFG, §4.3.1]), we can reformulate this as

Corollary 3.10.2:

$$H^q(\mathcal{G}, \mathrm{ind}_{\mathcal{H}}^{\mathcal{G}} M) \cong H^q(\mathcal{H}, M) \quad \text{for all } q \geq 0.$$

§3.11. Proof of One generator theorem, Step 0.

Since $\mathbb{T}$ is free of finite rank over Λ, if $\mathbb{T}$ is generated by one element $\Theta \in \mathfrak{m}_{\mathbb{T}}$ over Λ, the multiplication by Θ on $\mathbb{T}$ has its characteristic polynomial $D(X)$ of degree $e = \mathrm{rank}_{\Lambda} \mathbb{T}$ which is a distinguished polynomial with respect to $\mathfrak{m}_{\Lambda}$ satisfying $\mathbb{T} = \Lambda[[X]]/(D(X))$. Here a polynomial $f(X) \in A[X]$ is *distinguished* with respect to a prime A-ideal P if $f(X) \equiv X^{\deg(f)} \mod P$.

Since $\mathbb{T}$ is generated by $\mathrm{Tr}(\rho_{\mathbb{T}})$, the morphism $\pi : R^{ord} \to \mathbb{T}$ with $\pi \circ \boldsymbol{\rho} \cong \rho_{\mathbb{T}}$ is surjective. Thus we need to prove that R^{ord} is generated by one element over Λ. In other words, we prove that $t^*_{R^{ord}/\Lambda} := \mathfrak{m}_{R^{ord}}/(\mathfrak{m}^2_{R^{ord}} + \mathfrak{m}_{\Lambda}) \cong \mathrm{Sel}(Ad(\overline{\rho}))^{\vee}$ has dimension ≤ 1 over $\mathbb{F}$.

§3.12. Step 1: Restriction.

Write $\overline{G} = \mathrm{Gal}(F(Ad(\overline{\rho}))/\mathbb{Q})$. If $p \nmid |\overline{G}|$, plainly $H^1(\overline{G}, Ad(\overline{\rho})^*) = 0$. Otherwise, by Dickson's classification §2.12, $\overline{G}$ is isomorphic to either $\mathrm{PSL}_2(\mathbb{F}')$, $\mathrm{PGL}_2(\mathbb{F}')$ for a subfield $\mathbb{F}'$ of $\mathbb{F}$ or A_5 (when $p = 3$), and we know $H^1(\overline{G}, Ad(\overline{\rho})^*) = 0$ (e.g., Wiles' FLT paper Proposition 1.11). By restriction, for $M = F(Ad(\overline{\rho}))$, we find

$$\mathrm{Sel}(Ad(\overline{\rho})) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(G_M, Ad(\overline{\rho})^*).$$

This map has image in $\mathrm{Sel}_M(Ad(\overline{\rho}))$: $\mathrm{Sel}(Ad(\overline{\rho})) \hookrightarrow \mathrm{Sel}_M(Ad(\overline{\rho}))$. Thus we need to show $\dim_{\mathbb{F}} \mathrm{Sel}_M(Ad(\overline{\rho})) \leq 1$ under our assumptions.

Some notation: Let $\mathcal{O}$ be the integer ring of M, $\mathcal{O}_p = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\widehat{\mathcal{O}_p^\times} = \varprojlim_n \mathcal{O}_p^\times/(\mathcal{O}_p^\times)^{p^n}$ (the maximal p-profinite quotient of $\mathcal{O}_p^\times$). Similarly set $\widehat{\mathcal{O}_\wp^\times} = \varprojlim_n \mathcal{O}_\wp^\times/(\mathcal{O}_\wp^\times)^{p^n}$ for each prime factor $\wp | p$.

§**3.13. Step 2: Selmer sequence.** We fix $\wp_0$ and choose the inertia group I_0 at $\wp_0$ of G_M so that $\rho_{\mathbb{T}}|_{I_0}$ has values in upper triangular subgroup with the trivial quotient. For each $\wp|p$, we pick $g_\wp \in G$ and put $I_\wp := g_\wp I_0 g_\wp^{-1} \subset G_M$ is a inertia subgroup of $\wp$. By class field theory, writing G_M^{ab} for the maximal abelian quotient of G_M, $\widehat{\mathcal{O}_p^\times} \to G_M^{ab} \twoheadrightarrow C_M$ for $C_M := Cl_M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is exact, and applying $\mathrm{Hom}_{\mathbb{Z}[\overline{G}]}(?, Ad(\overline{\rho}))$,

$$\mathrm{Hom}_{\mathbb{Z}[\overline{G}]}(C_M, Ad(\overline{\rho})) \hookrightarrow \mathrm{Sel}_M(Ad(\overline{\rho}))^{\overline{G}} \xrightarrow{\pi} \mathrm{Hom}_{\mathbb{Z}_p[\overline{G}]}(\widehat{\mathcal{O}_p^\times}, Ad(\overline{\rho}))$$

with $\mathrm{Im}(\pi)$ made of ramified Selmer cocycles at p. Therefore, identifying the image of I_0 in G_M^{ab} with $\widehat{\mathcal{O}_{\wp_0}^\times}$ by class field theory, $\phi \in \pi(\mathrm{Sel}_M(Ad(\overline{\rho}))^{\overline{G}})$ has values over $\widehat{\mathcal{O}_{\wp_0}^\times}$ in the upper nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{sl}_2(\mathbb{F})$ and in $Ad(\overline{\rho}(g_\wp))(\mathfrak{n}) = g_\wp \mathfrak{n} g_\wp^{-1}$ over $\widehat{\mathcal{O}_\wp^\times}$.

Let $\overline{D}$ be the p-decomposition subgroup $\overline{D} \subset \overline{G}$ of $\wp_0$.

§**3.14. Step 3: Use of Shapiro's lemma.** Note $p \nmid |\overline{D}|$. Then the isomorphism class of a p-torsion-free $\mathbb{Z}_p[\overline{D}]$-module L of finite type is determined by the isomorphism class of $\mathbb{Q}_p[\overline{D}]$-modules $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. By p-adic logarithm and the normal basis theorem, $\widehat{\mathcal{O}_\wp^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathrm{Ind}_1^{\overline{D}} \mathbb{Q}_p = \mathbb{Q}_p[\overline{D}]$ as $\mathbb{Q}_p[\overline{D}]$-modules. We conclude $\widehat{\mathcal{O}_\wp^\times} \cong \mu_p(M_\wp) \oplus \mathrm{Ind}_1^{\overline{D}} \mathbb{Z}_p$. Up to p-torsion, the p-profinite completion $\widehat{\mathcal{O}_p^\times}$ is isomorphic to $\mathbb{Z}_p[\overline{G}] = \mathrm{Ind}_1^{\overline{G}} \mathbb{Z}_p$. If $\mu_p(M_\wp) = \{1\}$, we get

$$\mathrm{Hom}_{\mathbb{Z}_p[\overline{G}]}(\widehat{\mathcal{O}_p^\times}, Ad(\overline{\rho})) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p, Ad(\overline{\rho})) \cong Ad(\overline{\rho})$$

from Shapiro's lemma in which $\pi(\mathrm{Sel}_M(Ad(\overline{\rho}))^{\overline{G}})$ is sent into $\mathfrak{n}$ having dimension 1 over $\mathbb{F}$. Thus the theorem follows from our assumption: $\mathrm{Hom}_{\mathbb{Z}[\overline{G}]}(Cl_M, Ad(\overline{\rho})) = Cl_M \otimes_{\mathbb{Z}[\overline{G}]} Ad(\overline{\rho}) = 0$; so, $\dim_{\mathbb{F}} \mathrm{Sel}_M(Ad(\overline{\rho}))^{\overline{G}} = \dim_{\mathbb{F}} \pi(\mathrm{Sel}_M(Ad(\overline{\rho}))^{\overline{G}}) \leq 1$. This finishes the proof when $\mu_p(M_\wp) = \{1\}$.

§3.15. Final step: $\mu_p(M_{\wp_0}) \neq \{1\}$.

Now assume that $\mu_p(M_{\wp_0})$ has order p. By our assumption, $Ad(\overline{\rho})|_{\overline{D}}$ does not contain $\overline{\omega}$ for $\overline{\omega} := \nu_p \mod p\mathbb{Z}_p$. We have $\widehat{\mathcal{O}_p^\times} \cong \mathrm{Ind}_{\overline{D}}^{\overline{G}} \mu_p(\overline{\mathbb{Q}}) \oplus \mathrm{Ind}_1^{\overline{G}} \mathbb{Z}_p$, since $\widehat{\mathcal{O}_{\wp_0}^\times} \cong \mu_p(M_{\wp_0}) \oplus \mathrm{Ind}_1^{\overline{D}} \mathbb{Z}_p$. Since $Ad(\overline{\rho})|_{\overline{D}}$ does not contain $\overline{\omega}$, by Shapiro's lemma,

$$\mathrm{Ind}_{\overline{D}}^{\overline{G}} \mu_p(M_{\wp_0}) \otimes_{\mathbb{Z}[\overline{G}]} Ad(\overline{\rho}) = 0,$$

and we find

$$\mathrm{Hom}_{\mathbb{Z}_p[\overline{G}]}(\widehat{\mathcal{O}_p^\times}, Ad(\overline{\rho})) \cong \mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathrm{Ind}_1^{\overline{G}} \mathbb{Z}_p, Ad(\overline{\rho})).$$

Then by the same argument as above, we conclude

$$\dim_{\mathbb{F}} \mathrm{Sel}_M(Ad(\overline{\rho}))^{\overline{G}} = \dim_{\mathbb{F}} \pi(\mathrm{Sel}_M(Ad(\overline{\rho}))^{\overline{G}}) \leq 1$$

as desired. $\square$

§3.16. Weight 1 cyclicity theorem.

Corollary 3.16: *Suppose that f has weight 1, $p \nmid |\mathrm{Im}(\rho_f)|$ (or p is tamely ramified in $F(\rho_f)$), $\mathbb{T} = R^{ord}$, $Ad(\overline{\rho}) \otimes_{\mathbb{Z}[D_p]} \mu_p(M_{\wp_0}) = 0$ and $Cl_M \otimes_{\mathbb{Z}[G]} Ad(\rho_f) = 0$ for $M := F(Ad(\overline{\rho}))$. Then if $A \in CL_\Lambda$, $\mathrm{Sel}(Ad(\rho_A))^\vee \cong A/L_\Lambda(\varphi)A$ ($L_\Lambda(\varphi) := \varphi(L_\Lambda)$) as A-modules for each $\varphi \in \mathrm{Hom}_{CL_\Lambda}(R^{ord}, A)$ for $L_\Lambda \in \mathbb{T}$ in Theorem 2.24.*

Proof. By theorem 3.4 and the second fundamental exact sequence

$$\mathbb{T} \cong (D)/(D^2) \xrightarrow{x \mapsto L_\Lambda x} \mathbb{T} \cong \mathbb{T}dX \to \Omega_{\mathbb{T}/\Lambda} \to 0.$$

Tensoring A, we get $\mathrm{Sel}(Ad(\rho_A))^\vee \cong \Omega_{\mathbb{T}/\Lambda} \otimes_{\mathbb{T}} A \cong A/L_\Lambda(\varphi)A$ by Theorem 1.27 as desired. □

§3.17. Open questions:

If f has weight $k \geq 2$ which is not a binary theta series of an imaginary quadratic field, we have $R^{ord}_{\mathfrak{p}} = \mathbb{T}_{\mathfrak{p}} = \Lambda$ for almost all ordinary primes $\mathfrak{p}$ of $\mathbb{Z}[f]$; so, Sel$(Ad(\overline{\rho}_{\mathfrak{p}})) = 0$ for almost all ordinary $\mathfrak{p}$. Note Im$(\overline{\rho}_{\mathfrak{p}})$ modulo center is isomorphic to PGL$_2(\mathbb{F}')$ or PSL$_2(\mathbb{F}')$ (for $\mathbb{F}' \subset \mathbb{F}$) by a result of Ribet. For these classical groups, group theorists proved $H^1(\overline{G}, Ad(\overline{\rho})) = 0$, but $H^2(\overline{G}, Ad(\overline{\rho}))$ is 1-dimensional. Therefore as seen in §3.12, for $M = F(Ad(\overline{\rho}_{\mathfrak{p}}))$, Sel$(Ad(\overline{\rho}_{\mathfrak{p}})) \subset$ Sel$_M(Ad(\overline{\rho}_{\mathfrak{p}}))^{\overline{G}}$, which may not be surjective. Is this an isomorphism? If so, by the exact sequence in §3.13, $Cl_M \otimes_{\mathbb{Z}[G]} Ad(\overline{\rho}_{\mathfrak{p}}) = 0$. Is this vanishing of the $Ad(\overline{\rho}_{\mathfrak{p}})$-part of CL_M true for almost all $\mathfrak{p}$? In the 1-dimensional case, for $\overline{\omega}_p = (\nu_p \bmod p\mathbb{Z}_p)$, if we fix an integer $k > 0$,

$$Cl_{\mathbb{Q}[\mu_p]} \otimes_{Z[\overline{G}]} \overline{\omega}_p^{1-2k} = 0 \Leftrightarrow p \nmid \zeta(1-2k)$$

by Herbrand-Ribet theorem. Kummer–Vandiver conjecture (true for p up to 2 billion) tells us $Cl_{\mathbb{Q}[\mu_p]} \otimes_{Z[\overline{G}]} \overline{\omega}_p^{2k} = 0?$ for all p.

§3.18. Induced representation from a real quadratic field. We now study a very specific case of weight 1 which produces $\mathbb{T}_{\mathfrak{p}} \neq \Lambda$ for all ordinary minimal $\mathfrak{p}$. Fix a real quadratic field $F = \mathbb{Q}[\sqrt{D}]$ with discriminant $D > 0$ and integer ring O and a finite order character $\varphi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{Z}[\varphi]^{\times}$. Write $\mathfrak{f}$ for the conductor of φ. By class field theory, we may regard $\varphi : Cl_F^{+}(\mathfrak{f}) \to \mathbb{Z}[\varphi]^{\times}$. Assume that $\varphi(\xi) = -1$ for any totally negative $\xi \in O$ with $\xi \equiv 1 \mod \mathfrak{f}$ (i.e., φ ramifies at one infinite place ∞ of F). By Hecke, $f = \sum_{0 \neq \mathfrak{a} : O\text{-ideals}} \varphi(\mathfrak{a}) q^{N(\mathfrak{a})}$ is in $S_1(C, \varphi|_{\mathbb{Z}}\alpha)$ for $\alpha := \left(\frac{D}{}\right)$. f is a primitive form of conductor $C = D \cdot N(\mathfrak{f})$.

The associated Galois representation is given by the Artin representation $\rho_f = \mathrm{Ind}_F^{\mathbb{Q}} \varphi$. Indeed, by the explicit form of induced representation, $\mathrm{Tr}(\rho_f(\mathrm{Frob}_l)) = 0 = a(l, f)$ if a prime l is inert in F and $\mathrm{Tr}(\rho_f(\mathrm{Frob}_l)) = \varphi(\mathfrak{l}) + \varphi(\mathfrak{l}^{\varsigma}) = a(l, f)$ if $(l) = \mathfrak{l}\mathfrak{l}^{\varsigma}$ with $\mathfrak{l} \neq \mathfrak{l}^{\varsigma}$ for the non-trivial automorphism ς of F.

§3.19. Irreducibility. Let $\overline{\varphi} = \overline{\varphi}_{\mathfrak{p}} := (\varphi \mod \mathfrak{p})$. Thus $\overline{\rho} = \overline{\rho}_{\mathfrak{p}} = \mathrm{Ind}_F^{\mathbb{Q}} \overline{\varphi}$. Since $\overline{\varphi}$ has order prime to p, $\overline{\rho}_{\mathfrak{p}}$ is minimal. Suppose $\overline{\varphi} \neq \overline{\varphi}_{\varsigma}$ with $\varphi_{\varsigma}(g) = \varphi(\varsigma^{-1} g \varsigma)$ for $\varsigma \in G$ inducing non-trivial automorphism of F. Then for $H = \mathrm{Ker}(\alpha : G \to \{\pm 1\})$, the normalizer of $\overline{\rho}_{\mathfrak{p}}(G)$ contains $\begin{pmatrix} 0 & \overline{\varphi}(\varsigma^2) \\ 1 & 0 \end{pmatrix}$ and a diagonal matrix with distinct diagonal entries; so, the centralizer of $\overline{\rho}_{\mathfrak{p}}$ is made of scalar matirx. Thus we have shown the "$\Rightarrow$"-direction of

Lemma 3.19: *We have $\overline{\varphi} \neq \overline{\varphi}_{\varsigma}$ if and only if $\overline{\rho}_{\mathfrak{p}} = \mathrm{Ind}_F^{\mathbb{Q}} \overline{\varphi}$ is absolutely irreducible.*

If $\overline{\varphi} = \overline{\varphi}_{\varsigma}$, the centralizer of $\mathrm{Ind}_F^{\mathbb{Q}} \overline{\varphi}$ contains also an anti-diagonal element and hence it is bigger than the center, showing $\mathrm{Ind}_F^{\mathbb{Q}} \overline{\varphi}$ is reducible. Hereafter we assume $\overline{\varphi} \neq \overline{\varphi}_{\varsigma}$ or equivalently $\overline{\varphi}^{-} := \overline{\varphi}\overline{\varphi}_{\varsigma}^{-1} \neq 1$.

§3.20. Criterion for inducedness.

Theorem 3.20 *For a representation $\rho_A : G \to \mathrm{GL}_2(A) \in \mathcal{D}(A)$ for $A \in CL_B$, suppose ρ mod $\mathfrak{m}_A$ is absolutely irreducible. Then $\rho \otimes \alpha \cong \rho$ if and only if $\rho = \mathrm{Ind}_{\mathcal{H}}^{G} \phi$ for a character $\phi : \mathcal{H} \to A^\times$.*

Proof of $\Leftarrow$: By the explicit form ($G = \mathcal{H} \sqcup \mathcal{H}\varsigma$):

$$\rho(g) = \mathrm{Ind}_{\mathcal{H}}^{G} \phi(g) = \begin{pmatrix} \phi(g) & \phi(g\varsigma) \\ \phi(\varsigma^{-1}g) & \phi(\varsigma^{-1}g\varsigma) \end{pmatrix},$$

for $j = \mathrm{diag}[-1, 1]$, we find $\rho \cdot \alpha = j\rho j^{-1}$, where $\rho \cdot \alpha(g)$ is the matrix $\rho(g)$ multiplied by the scalar $\alpha(g)$. $\square$

§3.21. Converse.

Conversely, suppose $\rho \cdot \alpha = J \rho J^{-1}$. Then J^2 commutes with ρ as $\alpha^2 = 1$. Let $E \subset M_2(A)$ be the subalgebra generated over A by $\rho(g) \in G$. By absolute irreduciblity of ρ mod $\mathfrak{m}_A$ (Lemma 3.19), $E \otimes_A A/\mathfrak{m}_A = M_2(\mathbb{F})$; so, by Nakayama's lemma, $E = M_2(A)$. Thus J^2 is a scalar. Since $\rho \cdot \alpha \neq \rho$, J is non-scalar.

Since $\overline{\rho}_{\mathfrak{p}} \cdot \alpha = j \overline{\rho}_{\mathfrak{p}} j^{-1}$, J mod $\mathfrak{m}_A = \overline{z} \cdot j$ mod $\mathfrak{m}_A$ for $\overline{z} \in \mathbb{F}^{\times}$. Lifting $\overline{z}$ to $z \in A^{\times}$ and replacing J by $z^{-1}j$, we may assume that $J \equiv j \mod \mathfrak{m}_A$. This implies $z := J^2 \in 1 + \mathfrak{m}_A$ and hence the scalar z is a square in A by $p > 2$. Thus replacing J by $\sqrt{z}^{-1} J$, we may assume that J has eigenvalues ± 1. The -1-eigenspace of J is stable under H giving rise to a character $\phi : H \to A^{\times}$ with ϕ mod $\mathfrak{m}_A = \overline{\varphi}$. Then by Shapiro's lemma, we find $\rho = \mathrm{Ind}_F^{\mathbb{Q}} \phi$. $\square$

§3.22. Minimality and ordinarity of $\rho = \rho_{\mathfrak{p}} := \mathrm{Ind}_F^{\mathbb{Q}} \varphi$.
By minimality, for the conductor $\mathfrak{f}$ of φ, $\overline{\rho}$ has conductor $C := N(\mathfrak{f})D$. We pick a prime $\wp | p$ of F.

(R_-) If $p|C$ and $\alpha(p) = -1$, $\overline{\varphi}$ and $\overline{\varphi}_\varsigma$ both ramifies at p; so, no p-unramified quotient of $\overline{\rho}|_{D_p}$ (No ordinary case).
(R_+) If $p|C$ and $\alpha(p) = 1$ (so, $(p) = \wp\wp^\varsigma$ with $\wp \neq \wp^\varsigma$), to have p-unramified quotient, we may assume that $\wp|\mathfrak{f}$ and $\wp^\varsigma \nmid \mathfrak{f}$. In this case $\delta = \varphi_\varsigma$ (p-distinguished).
(D) If $p|C$ and $\alpha(p) = 0$ (so, $(p) = \wp^2$). To have p-unramified quotient, $\wp \nmid \mathfrak{f}$. We need to have $\delta(\mathrm{Frob}_p) = \varphi(\mathrm{Frob}_\wp)$ and $\rho|_{D_p} = \mathrm{diag}[\alpha\delta, \delta]$ (p-distinguished).
($U_\pm$) If $\alpha(p) = \pm 1$ and $p \nmid C$, ρ and $\overline{\rho}$ is unramified. We choose a prime factor $\wp | p$.

§3.23. Choice of δ and $\overline{\delta}$ in Cases D and (U$_\pm$).

In Case U$_+$, $\rho(\mathrm{Frob}_p) = \mathrm{diag}[\varphi(\wp), \varphi_\varsigma(\wp)]$ and we take $\overline{\delta} := (\varphi_\varsigma \bmod \mathfrak{p})$ (in this case, $D_p = D_{\mathfrak{p}}$) and $\delta = \varphi_\varsigma$.

In Case U$_-$, taking $\varsigma = \mathrm{Frob}_p$ and choosing a square root $\delta = \delta(\mathrm{Frob}_p)$ of $\varphi(\mathrm{Frob}_p^2)$, we have $\rho(\mathrm{Frob}_p) = \begin{pmatrix} 0 & \delta^2 \\ 1 & 0 \end{pmatrix}$ and $\rho(\mathrm{Frob}_p) \sim \mathrm{diag}[\delta, -\delta]$ (automatically p-distinguished). We choose an unramified character $\delta : D_p/I_p = \langle \mathrm{Frob}_p \rangle \to \overline{\mathbb{Q}}^\times$ such that $\delta^2 = \varphi|_{D_p}$ and $\delta(\mathrm{Frob}_p) = \delta$. Put $\overline{\delta} = (\delta \bmod \mathfrak{p})$.

In Case D, φ is unramified at p, and hence $\varphi|_{D_p} = \varphi_\varsigma|_{D_p}$. Then $\rho|_{D_p}$ is a direct sum of subspaces on which I_p acts trivially and by α. The action of D_p on $H^0(I_p, \mathrm{Ind}_F^{\mathbb{Q}} \varphi)$ gives a character $\delta : D \to \mathbb{Z}[\varphi]^\times$ and $\rho|_{D_p} = \mathrm{diag}[\delta\alpha, \delta]$ (p-distingulshed).

§3.24. p-stabilization in Case $\mathbf{U}_{\pm}$.

If $g \in S_k(N,\psi)$ is a Hecke eigenform for N prime to p, writing $\lambda_g(T(p)) = \alpha + \beta$ and $\alpha\beta = \chi(p)$, define $g_\alpha(z) = g(z) - \beta g(pz)$. Then g is a Hecke eigenform of level Np with $g_\alpha|U(p) = \alpha \cdot g_\alpha$. The form g_α is called the p-stabilization of g with $U(p)$-eigenvalues α.

Let $\beta = \varphi(\mathfrak{p})$ if $\alpha(p) = 1$ and $\beta = -\delta(\mathrm{Frob}_p)$ if $\alpha(p) = -1$. Replacing f by $f^{ord} := f_\alpha$ $(\alpha\beta = \psi(p))$, we have $f^{ord}|U(p) = \delta(\mathrm{Frob}_p) f^{ord}$. Hereafter we choose the p-stabilized form f^{ord} in place of f (and write it f). Thus δ has values in $\mathbb{Z}[f]^{\times}$, and write the level of f as N (so, $N = C$ if $p|C$ and $N = Cp$ otherwise). Put $\overline{\rho}_{\mathfrak{p}} := \rho_f \mod \mathfrak{p}$. If $\alpha(p) = 1$, p-distinguishedness $\Leftrightarrow \overline{\varphi}^{-}|_{D_p} \neq 1$ for $\overline{\varphi}^{-} = \overline{\varphi\varphi_{\varsigma}^{-1}}$.

§3.25. An involution of $\mathcal{D}$ in Case U$_+$.

Take $\overline{\rho} = \mathrm{Ind}_F^{\mathbb{Q}} \overline{\varphi}$ in the matrix form in §3.6 (1). Thus

$$j(\overline{\rho} \cdot \alpha) j^{-1} = \overline{\rho}.$$

In Case U$_+$, define $\sigma : \mathcal{D}(A) \to \mathcal{D}(A)$ by $\sigma([\rho_A]) = [j(\rho_A \cdot \alpha)j^{-1}]$ which is an automorphism of the functor $\mathcal{D}$ inducing an involution $\sigma \in \mathrm{Aut}(R^{ord}/\Lambda)$.

Since $\det(\rho_A \cdot \alpha) = \det(\rho_A)$, this operation does not affect the determinant character and the ordinary character δ_A; so, σ induces an involution on R_χ. Define "$\pm$"-eigenspaces

$$R_\chi^{\pm} := \{x \in R_\chi | \sigma(x) = \pm x\} \quad \text{and} \quad R_\pm^{ord} \subset R^{ord}.$$

§3.26. No involution of $\mathcal{D}$ in Cases U$_-$ and D.

In Cases U$_-$ and D, by non-triviality of $\alpha|_{D_p}$, $j(\overline{\rho}\cdot\alpha)j^{-1}$ has specified ordinary character $\overline{\delta}\chi$, which violates ordinarity. So we assume $\alpha(p) = 1$ hereafter; i.e., we are in Case U$_+$.

Let $\mathfrak{d} := R^{ord}(\sigma - 1)R^{ord}$. Then $R^{ord}/\mathfrak{d}$ is the maximal quotient of R^{ord} on which σ acts trivially. This means for $\rho := (\boldsymbol{\rho}^{ord} \bmod \mathfrak{d})$, $\rho \otimes \alpha \cong \sigma \circ \rho = \rho \Leftrightarrow \rho = \mathrm{Ind}_F^{\mathbb{Q}} \Phi$ for a character $\Phi : H \to R^{ord}/\mathfrak{d}$ for $H := \mathrm{Ker}(G \xrightarrow{\alpha} \{\pm\})$. What is Φ? Next goal is to know Φ.

Define $C_p := Cl_F^+(\wp^\infty) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ if $\alpha(p) = 1$. Note that $O_\wp^\times/\overline{\varepsilon^{\mathbb{Z}}}$ is finite for the totally positive fundamental unit ε of O and hence C_p is a finite p-group. If $p \nmid h_F$, then $C_p = \widehat{O}_\wp^\times/\varepsilon^{(p-1)\mathbb{Z}_p} \cong \Gamma/\Gamma^{\log_p(\varepsilon)/\log_p(1+p)}$; so, $W[C_p] \cong \Lambda/(\langle\varepsilon\rangle - 1)$ for $\langle\varepsilon\rangle = t^{\log_p(\varepsilon)/\log_p(\gamma)}$.

§3.27. Φ is a minimal universal character.

Theorem 3.27: *Assume* $\mathrm{Ind}_F^{\mathbb{Q}}\varphi$ *is minimal (i.e., the order of* φ *is prime to* p*). We have* $R^{ord}/\mathfrak{d} \cong W[C_p]$ *for* C_p *and* $\Phi(h) = \varphi(h)h|_{H_\wp}$ *for the* $\wp$ *ray class field* $H_\wp$ *with* $\mathrm{Gal}(H_\wp/F) \cong C_p$*.*

Proof: If $\phi : \mathrm{Gal}(H_\wp/F) \to A^\times$ with $\phi \bmod \mathfrak{m}_A = \overline{\varphi}$, plainly $\mathrm{Ind}_F^{\mathbb{Q}}\phi \in \mathcal{D}(A)$. Thus $\mathrm{Ind}_F^{\mathbb{Q}}\Phi \in \mathcal{D}(W[C_p])$ and we have the universal map $\pi : R^{ord} \twoheadrightarrow W[C_p]$ with $\pi \circ \boldsymbol{\rho} \sim \mathrm{Ind}_F^{\mathbb{Q}}\Phi$. Since we have $j(\mathrm{Ind}_F^{\mathbb{Q}}\Phi \cdot \alpha)j^{-1} = \mathrm{Ind}_F^{\mathbb{Q}}\Phi$, σ acts trivially on $R^{ord}/\mathrm{Ker}(\pi)$ and hence $\mathfrak{d} \subset \mathrm{Ker}(\pi)$. Since σ acts trivially on $R^{ord}/\mathfrak{d}$, $\boldsymbol{\rho}^{ord} \bmod \mathfrak{d} \sim \mathrm{Ind}_F^{\mathbb{Q}}\Psi$ for character $\Psi : H \to R^{ord}/\mathfrak{d}$ (§3.20). By ordinarity, $\Psi_\varsigma\varphi_\zeta^{-1}$ is unramified over $D_\wp$. Since p is split, $\Psi\varphi^{-1}$ can ramify only at $\wp$. Thus $\Psi\varphi^{-1}$ factors through C_p. By the universality of Φ, we have $\pi' : W[C_p] \twoheadrightarrow R^{ord}/\mathfrak{d}$ such that $\Psi = \pi' \circ \Phi$. This map π' is onto because R^{ord} is generated by trace of $\boldsymbol{\rho}$. Then $\pi \circ \pi'$ is onto, and comparing the W-rank, we get $R^{ord}/\mathfrak{d} \cong W[C_p]$. □

§3.28. Corollary: $R^{ord} \neq R^{ord}_{+}, R_{\chi} \neq R^{+}_{\chi}$ **and** $\Omega_{R^{ord}/\Lambda} \neq 0$**.**

Proof: If $R^{ord}_{+} = R^{ord}$, then $\mathfrak{d} = 0$ and hence $R^{ord} = W[C_p]$. This is impossible as R^{ord} surjects down to $\mathbb{T}_{\mathfrak{p}}$ which has infinite rank over W. As we have seen, if $\Omega_{A/B} = 0$, $B \twoheadrightarrow A$. Thus $R^{ord}_{+} \neq R^{ord}$ implies $\Omega_{R^{ord}/R^{ord}_{+}} \neq 0$. By the fundamental exact sequence, $\Omega_{R^{ord}/\Lambda}$ surjects down to $\Omega_{R^{ord}/R^{ord}_{+}}$; so, $\Omega_{R^{ord}/\Lambda} \neq 0$.

Recall $R_{\chi} = R^{ord}/(t - \chi(\gamma))R^{ord} = R^{ord} \otimes_{\Lambda} \Lambda/(t - \chi(\gamma))$. Taking "+"-eigenspace of this identity, we get $R^{+}_{\chi} = R^{ord}_{+} \otimes_{\Lambda} \Lambda/(t - \chi(\gamma))$. Tensoring $\Lambda/(t - \chi(\gamma))$ over Λ with the exact sequence $R^{ord}_{+} \to R^{ord} \to C \to 0$ with $C \neq 0$, we find $R^{+}_{\chi} \to R_{\chi} \to C/(t - \chi(\gamma))C \to 0$. By Nakayama's lemma, $C = 0 \Leftrightarrow C/(t - \chi(\gamma))C = 0$. Thus $R^{+}_{\chi} \neq R_{\chi}$. □

§3.29. For which $\mathfrak{p}$, $R_{\mathfrak{p}}^{ord} = \mathbb{T}_{\mathfrak{p}}$ is regular?

Assuming $\overline{\rho} = \mathrm{Ind}_F^{\mathbb{Q}}\overline{\varphi}$ is minimal p-ordinary, the next goal is to specify the single generator Θ of $\mathbb{T}_{\mathfrak{p}}$ and study the ring structure of $\mathbb{T}_{\mathfrak{p}}$. For example, we ask

when is $\mathbb{T}_{\mathfrak{p}}$ a regular local ring?

In this setting, writing Θ for a well chosen generator, $\mathbb{T}_{\mathfrak{p}}$ is a regular local ring if and only if $\mathbb{T}_{\mathfrak{p}} \cong W_{\mathfrak{p}}[[\Theta]] \cong W_{\mathfrak{p}}[[X]]$ (the one variable power series ring) by $\Theta \mapsto X$, but still $\Lambda = W_{\mathfrak{p}}[[T]] \subsetneq \mathbb{T}_{\mathfrak{p}}$.

The distribution of such primes is another question we like to ask. Are these primes infinitely many? Or even of density 1? We explore these questions in the next couple of weeks.