Lecture slide No.2 for Math 207c
Adjoint Selmer groups

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For a fixed Hecke eigenform f of weight $k \geq 2$, we have its $\mathfrak{p}$-adic Galois representation $\{\rho_{f,\mathfrak{p}}\}_{\mathfrak{p}}$ for $\mathfrak{p}$ running over primes of $\mathbb{Z}[f]$. Defining $\overline{\rho}_{\mathfrak{p}} = (\rho_{f,\mathfrak{p}} \bmod \mathfrak{p})$, we study dependence on $\mathfrak{p}$ of the universal rings $R^{ord}_{\mathfrak{p}}$ and $R_{\chi,\mathfrak{p}}$ representing the deformation functors $\mathcal{D} = \mathcal{D}_{\mathfrak{p}}, \mathcal{D}_{\chi} = \mathcal{D}_{\chi,\mathfrak{p}} : \mathcal{C} \to SETS$ of $\overline{\rho}_{\mathfrak{p}}$ defined in §0.22 (assuming $\overline{\rho}_{\mathfrak{p}}$ satisfies (ord$_p$)).

As before, we write I_l for the inertia group of the l-decomposition subgroup $D_l \subset G$ and $\chi = \nu_p^{k-1}\psi$ ($f \in S_k(N,\psi)$). We write S for the set of ramified primes $l \neq p$ of $\overline{\rho}$ such that $\overline{\rho}|_{I_l} \cong \overline{\epsilon}_l \oplus \overline{\delta}_l$. The conductor of a local or Dirichlet character ψ is written as $C(\psi)$.

§**2.1. Integral modular forms.** Let $\mathbb{Z}[\psi]$ be the subring of $\mathbb{C}$ generated by the values of ψ. For an algebra $\mathbb{Z}[\psi] \subset A \subset \mathbb{C}$, let

$$S_k(\Gamma_0(N), \psi; A) := \{f \in S_k(\Gamma_0(N), \psi) | a(n, f) \in A \text{ for all } n > 0\},$$

where $f(z) = \sum_{n=1}^{\infty} a(n, f)q^n$ with $q = \exp(2\pi iz)$. Often we write $S_k(N, \psi)_{/A}$ for $S_k(\Gamma_0(N), \psi; A)$. We then define

$$h_k(N, \psi)_{/A} = A[T(n)|n = 1, 2, \ldots] \subset \mathrm{End}_A(S_k(N, \psi)_{/A}).$$

These are A–modules of finite type and (see [MFG, §3.1.8–9])

$$S_k(N, \psi)_{/A} = S_k(N, \psi)_{/\mathbb{Z}[\psi]} \otimes_{\mathbb{Z}[\psi]} A,$$
$$\mathrm{Hom}_A(S_k(N, \psi)_{/A}, A) \cong h_k(N, \psi)_{/A},$$
$$\mathrm{Hom}_A(h_k(N, \psi)_{/A}, A) \xrightarrow[\sim]{i} S_k(N, \psi)_{/A}.$$

Here the duality between h_k and S_k is given by $\langle T, f \rangle = a(1, f|T)$. By the identity of Hecke: $\langle T(n), f \rangle = a(n, f)$, we have $i(\phi) = \sum_{n=1}^{\infty} \phi(T(n))q^n$.

§2.2. Known and unknown facts for $h_k(N,\psi)$.

• Define for general $\mathbb{Z}[\psi]$-algebra A,

$$S_k(N,\psi)_{/A} = S_k(N,\psi)_{/\mathbb{Z}[\psi]} \otimes_{\mathbb{Z}[\psi]} A, \;\; h_k(N,\psi)_{/A} = h_k(N,\psi)_{/\mathbb{Z}[\psi]} \otimes_{\mathbb{Z}[\psi]} A.$$

Then $h_k(N,\psi)_{/A} = A[T(n)|n = 1,\dots] \subset \mathrm{End}_A(S_k(N,\psi)_{/A}))$ and the duality statement holds without any modification.

• If $N = C(\psi)$, $h_k(N,\psi)_{/A}$ is reduced for any $\mathbb{Z}[\psi]$-domain A flat over $\mathbb{Z}[\psi]$. This follows from the theory of new/old forms (see Miyake's book "Modular forms" (Springer) §4.6 (in his language, new form is called primitive form). Actually this is still true if $N = N_0C(\psi)$ with square-free $N_0|C(\psi)$ (why?).

Conjecture: Suppose $k \geq 2$. If N is cube-free, $h_k(N,\psi)_{/A}$ is reduced for any $\mathbb{Z}[\psi]$-domain A flat over $\mathbb{Z}[\psi]$.

This is known if $k = 2$ (Coleman/Edixhoven) and $k = 3$ for N square-free (D. Ulmer). See Coleman/Edixhoven paper Math. Ann. **310**, 119–127 (1998) and Ulmer paper: IMRN No. 7 (1995)

§2.3. Hecke eigenform

A cusp form $f \in S_k(N,\psi)_{/\mathbb{C}}$ is called a Hecke eigenform if $f|T(n) = \lambda(T(n))f$ for all n with the eigenvalues $\lambda(T(n))$ and $a(1,f) = 1$. This fact is equivalent to $f = \sum_{n=1}^{\infty} \lambda(T(n))q^n$ (by duality between S_k and h_k). Fix such an eigenform $f \in S_k(N,\psi)$. Then we get an algebra homomorphism $\lambda : h = h_k(N,\psi)_{/\mathbb{Z}[f]} \to \mathbb{Z}[f]$. Pick a prime $\mathfrak{p}|p$ of $\mathbb{Z}[f]$ and let $W = W_\mathfrak{p} = \mathbb{Z}_p[f] = \mathbb{Z}[f]_\mathfrak{p}$. Since $h_\mathfrak{p} := h_k(N,\psi)_{/W}$ is free of finite rank over W, for any maximal ideal $\mathfrak{m}$ of h_k, we have $\mathfrak{m} \cap W = \mathfrak{m}_W$. Since $h_\mathfrak{p}/\mathfrak{m}_W h_\mathfrak{p}$ is a finite ring; so, there is only a finite many maximal ideal of $h_\mathfrak{p}$; so, $h_\mathfrak{p} = \prod_\mathfrak{m} \mathrm{h}_\mathfrak{m}$ for the $\mathfrak{m}$-adic completion $h_\mathfrak{m}$ which is a local ring. We pick the unique local ring factor $\mathbb{T}_{\chi,\mathfrak{p}}$ of $h_\mathfrak{p}$ through which λ factors. First we study how $\Omega_{\mathbb{T}_{\chi,\mathfrak{p}}/W}$ depends on $\mathfrak{p}$. The idea is to relate $\Omega_{\mathbb{T}_{\chi,\mathfrak{p}}/W}$ with $\Omega_{h/\mathbb{Z}[f]}$ which is independent of $\mathfrak{p}$.

§2.4. A summary of general properties of $d_A : A \to \Omega_{A/B}$.

Let A A_j, B' be B-algebras.

(d1) $\Omega_{A_1 \times A_2/B} \cong \Omega_{A_1/B} \oplus \Omega_{A_2/B}$ $(d_{A_1 \times A_2} = d_{A_1} + d_{A_2})$ as $Der_B(A_1 \times A_2, M) = Der_B(A_1, M) \oplus Der_B(A_2, M)$.

(d2) $\Omega_{S^{-1}A/B} \cong \Omega_{A/B} \otimes_A S^{-1}A$ $(d_{S^{-1}A} = d_A \otimes 1)$ for a multiplicative set $1 \in S \subset A$.

(d3) $\Omega_{A \otimes_B B'/B'} \cong \Omega_{A/B} \otimes_B B'$ $(d_{A \otimes B'} = d_A \otimes 1)$.

Suppose that A is a B-module of finite type.

(d4) $\Omega_{A/B} = 0$ if A is a separable extension of a field B.

Indeed, if A is a field, $A = B[X]/(f(X))$ with θ image of X in A. Then $\Omega_{A/B} = (A/f'(\theta)A)d\theta = 0$ as $f'(\theta) \neq 0$ □

(d5) $\Omega_{A/B}$ is a torsion B-module if B is an integral domain of characteristic 0 and A is reduced. This follows from (d1–2) and (d4) since $A \otimes_B \mathrm{Frac}(B) = \mathrm{Frac}(A) = K_1 \times \cdots \times K_r$ for separable extensions K_i. What happens if B has characteristic $p > 0$.

§2.5. Preliminary lemmas.

Lemma 2.5.1. *Suppose that h is reduced. Then $\Omega_{h/\mathbb{Z}[f]}$ is a finite module.*

By (d5), $\Omega_{h/\mathbb{Z}[f]}$ is a torsion $\mathbb{Z}[f]$-module of finite type; so, it is finite. □

Lemma 2.5.2. *We have $\Omega_{\mathbb{T}_{\chi,\mathfrak{p}}/W_{\mathfrak{p}}} = \Omega_{h/\mathbb{Z}[f]} \otimes_h \mathbb{T}_{\chi,\mathfrak{p}}$.*

Note $h_{\mathfrak{p}} = h \otimes_{\mathbb{Z}[f]} W_{\mathfrak{p}}$. Thus by (d3), $\Omega_{h_{\mathfrak{p}}/W_{\mathfrak{p}}} = \Omega_{h/\mathbb{Z}[f]} \otimes_{\mathbb{Z}[f]} W_{\mathfrak{p}}$. Since $h_{\mathfrak{p}} = \prod_{\mathfrak{m}} h_{\mathfrak{m}}$ and $\mathbb{T}_{\chi,\mathfrak{p}}$ is one of $h_{\mathfrak{p}}$; so, $\Omega_{h_{\mathfrak{p}}/W_{\mathfrak{p}}} = \bigoplus_{\mathfrak{m}} \Omega_{h_{\mathfrak{m}}/W_{\mathfrak{p}}}$. If $\mathbb{T}_{\chi,\mathfrak{p}} = h_{\mathfrak{M}}$, $\Omega_{\mathbb{T}_{\chi,\mathfrak{p}}/W_{\mathfrak{p}}} = \Omega_{h_{\mathfrak{M}}/W_{\mathfrak{p}}} = \Omega_{h_{\mathfrak{p}}/W_{\mathfrak{p}}} \otimes_{h_{\mathfrak{p}}} h_{\mathfrak{M}}$; so, we get the desired formula. □

§2.6. Consequence of vanishing of differentials.

Let $A \in CL_B$.

Lemma 2.6.1. *Suppose that A is a torsion-free B-algebra. Then $\Omega_{A/B} \otimes_A A/\mathfrak{a} = 0$ for a proper A-ideal $\mathfrak{a}$ if and only if $A = B$.*

Proof. We need to prove $\Omega_{A/B} \otimes_A A/\mathfrak{a} = 0 \Rightarrow A = B$. By Nakayama's lemma, we have $\Omega_{A/B} = 0 \Leftrightarrow \Omega_{A/B} \otimes_A \mathbb{F} = 0 \Leftrightarrow \Omega_{A/B} \otimes_A A/\mathfrak{a} = 0$. Thus we may assume that $\mathfrak{a} = \mathfrak{m}_A$. Thus we have $t^*_{A/B} := \mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_B) = \Omega_{A/B} \otimes_A \mathbb{F} = 0$, which implies that $i_B(\mathfrak{m}_B) = \mathfrak{m}_A$, and therefore by the argument in §1.5, we have a surjective B-algebra homomorphism $\pi : B \twoheadrightarrow A$. Thus torsion-freeness tells us $\mathrm{Ker}(\pi) = 0$, and hence $A = B$. □

§2.7. Theorem: $\mathbb{T}_{\chi,\mathfrak{p}} = W_{\mathfrak{p}}$ for almost all $\mathfrak{p}$.

We actually prove
Theorem 2.7: *Let* $\mathrm{Ann}(f)$ *be the annihilator of* $\Omega_{h/\mathbb{Z}[f]}$ *in* $\mathbb{Z}[f]$*. Then* $\mathrm{Ann}(f)$ *is a non-zero ideal of* $\mathbb{Z}[f]$*, and if* $\mathfrak{p} \nmid \mathrm{Ann}(f)$*, then* $\mathbb{T}_{\chi,\mathfrak{p}} = W_{\mathfrak{p}}$*.*

Proof. By Lemma 2.5.1, $\mathrm{Ann}(f)$ is a non-zero ideal of $\mathbb{Z}[f]$ (could be $\mathbb{Z}[f]$ itself). By Lemma 2.6.1, we need to show that $\mathfrak{p} \nmid \mathrm{Ann}(f)$, then $\Omega_{\mathbb{T}_{\chi,\mathfrak{p}}/W_{\mathfrak{p}}} = 0$. By Lemma 2.5.2,

$$\Omega_{\mathbb{T}_{\chi,\mathfrak{p}}/W_{\mathfrak{p}}} = \Omega_{h/\mathbb{Z}[f]} \otimes_{\mathbb{Z}[f]} \mathbb{T}_{\chi,\mathfrak{p}}. \qquad (*)$$

If $\mathfrak{p} \nmid \mathrm{Ann}(f)$, then $\mathbb{Z}[f] - \mathfrak{p}$ contains an element a which kill $\Omega_{h/\mathbb{Z}[f]}$ which is a unit in $\mathbb{T}_{\chi,\mathfrak{p}}$. Therefore the multiplication by a kills the right hand side of $(*)$ and is an automorphism of the left-hand-side; so, $\Omega_{\mathbb{T}_{\chi,\mathfrak{p}}/W_{\mathfrak{p}}} = 0$. □

§2.8. Old and new form.

For a modular form $g \in S_k(M, \varphi)_{/\mathbb{C}}$, $g(mz) = g|[m](z)$ for $0 < m \in \mathbb{Z}$ is in $S_k(Mm, \varphi)$. A linear combination in $S_k(N, \psi)$ of cusp forms of the form $g|[m]$ with $m > 1$ and g of lower level is called an old form. The orthogonal complement under Peterson inner product of the subspace of old forms is called the space of new forms. These spaces are stable under Hecke operators. A Hecke eigenform in $S_k(N, \psi)$ is called *primitive* if f is new. Among cusp forms of varying level with eigenvalues for $T(l)$ given by $\lambda(T(l))$ for almost all l, there exists a unique Hecke eigenform of minimal level C, and that is the primitive form. The level C is called the *conductor* of f. For all these, see Miyake's book "Modular forms" Chapter 4 (from Springer). Hereafter the fixed eigenform f is primitive of conductor C (so, $f \in S_k(C, \psi)$).

§2.9. Modular Galois representation. The cusp form f has a $\mathfrak{p}$-adic Galois representation $\rho_{\mathfrak{p}} = \rho_{f,\mathfrak{p}}$ with values in $GL_2(W_{\mathfrak{p}})$ for each prime $\mathfrak{p}$ of $\mathbb{Z}[f]$ satisfying (e.g., [GME, §4.2])

(G1) $\rho_{\mathfrak{p}}$ is unramified outside pC ($\mathfrak{p}|p$);

(G2) $\det(1 - \rho_{\mathfrak{p}}(\mathrm{Frob}_l)X) = 1 - \lambda(T(l))X + \chi(l)X^2$ for $l \nmid Cp$;

(G3) If $a(p,f) = \lambda(T(p)) \notin \mathfrak{p}$, $\rho_{\mathfrak{p}}|_{D_p} \cong \begin{pmatrix} \epsilon_p & * \\ 0 & \delta_p \end{pmatrix}$ (ordinarity);

Conjecture: $a(p,f) \notin \mathfrak{p}$ for density 1 primes $\mathfrak{p}$?

(G4) Writing the l-primary part of an integer $N > 0$ as N_l, if $C_l = C(\psi)_l$ for a prime $l|C$ ($l \neq p$), then $\rho_{\mathfrak{p}}|_{I_l} \cong \begin{pmatrix} \psi_l & 0 \\ 0 & 1 \end{pmatrix}$;

(G5) If $C_l = lC_l(\psi)$ ($l \neq p$), then $\rho_{\mathfrak{p}}|_{D_l} \cong \begin{pmatrix} \eta\nu_p & * \\ 0 & \eta \end{pmatrix}$ for a Galois character $\eta : D_l \to W_{\mathfrak{p}}^{\times}$ such that $\eta^2\nu_p = \chi_l$;

(G6) If $l^2|(C/C(\psi))$ ($l \neq p$), then $\lambda(T(l)) = 0$ and $\rho|_{D_l}$ is either absolutely irreducible or isomorphic to $\begin{pmatrix} \epsilon_l & 0 \\ 0 & \delta_l \end{pmatrix}$ with $C(\epsilon_l)C(\delta_l) = C_l$ with $C(\epsilon_l) > 1$ and $C(\delta_l) > 0$.

§2.10. Modular deformation.

Fix a $\mathbb{Z}[f]$-prime $\mathfrak{p}|p > 2$ and a primitive form $f \in S_k(C, \psi)$ of conductor C. Assume that $\rho_\lambda = \rho_f = \rho_{f,\mathfrak{p}}$ is minimal and satisfies (ord$_p$). Let $\overline{\rho} = \overline{\rho}_{f,\mathfrak{p}} := \rho_{\mathfrak{p}}$ mod $\mathfrak{p}$ (p-distinguished is satisfied by $\overline{\rho}$), and consider deformation functors $\mathcal{D} = \mathcal{D}_{\mathfrak{p}}, \mathcal{D}_\chi = \mathcal{D}_{\chi,\mathfrak{p}}$ for $\overline{\rho}$. Write $R_\chi = R_{\chi,\mathfrak{p}}$ (resp. $R^{ord} = R^{ord}_{\mathfrak{p}}$) for the universal ring representing $\mathcal{D}_{\chi,\mathfrak{p}}$ (resp. $\mathcal{D}_{\mathfrak{p}}$). Consider Tr$(\boldsymbol{\rho}_{\chi,\mathfrak{p}}) = \sum_{\lambda \in \mathrm{Hom}_{W_{\mathfrak{p}}\text{-alg}}(\mathbb{T}_{\chi,\mathfrak{p}}, \overline{\mathbb{Q}}_p)}$ Tr(ρ_λ). By (G2), Tr$(\rho_{\mathbb{T},\chi,\mathfrak{p}})$(Frob$_l$) $= T(l)|_{\mathbb{T}_{\chi,\mathfrak{p}}}$ for all primes $l \nmid Cp$. By Chebotarev density, Tr$(\rho_{\mathbb{T},\chi,\mathfrak{p}})$ has values in $\mathbb{T}_{\chi,\mathfrak{p}}$. By the theory of pseudo-character, we have a Galois representation $\rho_{\mathbb{T}} = \rho_{\mathbb{T},\chi,\mathfrak{p}} : G \to \mathrm{GL}_2(\mathbb{T}_{\chi,\mathfrak{p}})$ with Tr$(\rho_{\mathbb{T}})$(Frob$_l$) $= T(l)|_{\mathbb{T}_{\chi,\mathfrak{p}}}$. Since trace determines representation (if irreducible) over a field, we have $\rho_{\mathbb{T}} \in \mathcal{D}_\chi(\mathbb{T}_{\chi,\mathfrak{p}})$. Thus we have a universal map $\pi : R_{\chi,\mathfrak{p}} \to \mathbb{T}_{\chi,\mathfrak{p}}$ such that $\pi \circ \boldsymbol{\rho}_\chi \sim \rho_{\mathbb{T}}$.

§2.11. The $R = \mathbb{T}$ theorem. Here is a theorem of Taylor–Wiles proven in 1995, writing $p^* = (-1)^{(p-1)/2}p$ (see [MFG, §3.2.4]):

Theorem 2.11: *Assume that $\overline{\rho}$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{p^*}])$ is absolutely irreducible (Taylor–Wiles condition). If $k \geq 2$, π induces an isomorphism $R_{\chi,\mathfrak{p}} \cong \mathbb{T}_{\chi,\mathfrak{p}}$ identifying $\boldsymbol{\rho}_\chi$ with $\rho_{\mathbb{T},\chi,\mathfrak{p}}$. Moreover we have a presentation $\mathbb{T}_{\chi,\mathfrak{p}} \cong \frac{W_\mathfrak{p}[[T_1,\dots,T_r]]}{(S_1,\dots,S_r)}$ (a local complete intersection over $W_\mathfrak{p}$) for $r = \dim_\mathbb{F} t_{\mathbb{T}_{\chi,\mathfrak{p}}/W_\mathfrak{p}}$.*

• By Frobenius reciprocity law, the Taylor–Wiles condition fails $\Leftrightarrow \overline{\rho} \cong \mathrm{Ind}_{\mathbb{Q}[\sqrt{p^*}]}^{\mathbb{Q}} \varphi$ for a character $\varphi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{p^*}]) \to \mathbb{F}^\times$.
• This condition of irreducibility over k is mostly removed by Khare/Ramakrishna/Thorne/Kalyanswamy. See Kalyanswamy's thesis published in Mathematical Research Letters 25(4) (2016).
• It is known that $\mathbb{T}_{\chi,\mathfrak{p}}$ is reduced if the prime-to-p conductor of $\overline{\rho}$ match the prime-to-p level of f (e.g., under minimality).

§**2.12. Classification of** $\mathrm{Im}(\overline{\rho})$ **modulo center.** Leonard Eugene Dickson in his book: "Linear groups" (1901) in §260 gave a classification of subgroups $\mathcal{G} \subset \mathrm{PGL}_2(\mathbb{F})$ given by $\mathrm{Im}(\overline{\rho})$ modulo center:

(G) If $p \Big| |\mathcal{G}|$, $\mathcal{G}$ is conjugate to $\mathrm{PGL}_2(k)$ or $\mathrm{PSL}_2(k)$ for a subfield $k \subset \mathbb{F}$ as long as $p > 3$ (when $p = 3$, $\mathcal{G}$ can be A_5).

Suppose $p \nmid |\mathcal{G}|$ (so, $p \geq 5$). Then $\mathcal{G}$ is given as follows.

(C) $\mathcal{G}$ is cyclic ($\Rightarrow \mathrm{Im}(\overline{\rho})$ is abelian; $\overline{\rho}$ is reducible).

(D) $\mathcal{G}$ is isomorphic to a dihedral group D_a of order $2a$ (so, $\overline{\rho} = \mathrm{Ind}_K^{\mathbb{Q}} \overline{\varphi}$ for a quadratic field K), and $\mathbb{F} = \mathbb{F}_p[\overline{\varphi}]$ (the field generated by the values of $\overline{\varphi}$).

(E: Exceptional cases) $\mathcal{G}$ is either isomorphic to A_4, S_4 ($\mathbb{F} = \mathbb{F}_p$), or A_5 ($\mathbb{F} = \mathbb{F}_p$ if $p \equiv \pm 1, 0 \mod 5$ and $\mathbb{F}_{p^2}$ otherwise).

In Cases (G), (D), (E), $\overline{\rho}$ is absolutely irreducible, and in Cases (C) and (D) with $K = \mathbb{Q}[\sqrt{p^*}]$, Taylor–Wiles condition fails.

§2.13. If $k \geq 2$, $R_{\chi,\mathfrak{p}} = W_{\mathfrak{p}}$ and $r = 0$ for most ordinary $\mathfrak{p}$. By a result of Ribet, if f is not a binary theta series (i.e., a theta series of the norm form of a quadratic field), if $k \geq 2$, except for finitely many $\mathfrak{p}$, $\overline{\rho}_{f,\mathfrak{p}}$ falls in Case G; so, it satisfies Taylor–Wiles condition. Thus the assertion of the section title follows from Theorem 2.7 (and the $R = \mathbb{T}$ theorem).

If f mod $\mathfrak{p}$ is a theta series associated to a quadratic field K and $k \geq 2$, unless $K = \mathbb{Q}[\sqrt{p^*}]$, the same outcome.

If $k = 1$, under irreducibility, we are either in Case (D) or (E) and $\rho = \rho_{f,\mathfrak{p}}$ is independent of $\mathfrak{p}$ (or in short, ρ has finite image and has values in $\mathrm{GL}_2(\mathbb{Z}[f])$ for a finite extension $\mathbb{Z}[f]$ generated by the values of $\mathrm{Tr}(\rho)$ over $\mathbb{Z}$. We do not know the distribution of primes $\mathfrak{p}$ with $R_{\chi,\mathfrak{p}} = W_{\mathfrak{p}}$ except when $\rho = \mathrm{Ind}_K^{\mathbb{Q}} \varphi$ for real quadratic K. We study real quadratic case later. Next goal is to study this question for $B = \Lambda$. We ask if $R^{ord} = \Lambda$ for most $\mathfrak{p}$?

§**2.14. Definition of "big" $\mathbb{T}_{\mathfrak{p}}$.** Start with $\overline{\rho} = \rho_{f_0,\mathfrak{p}} \mod \mathfrak{p}$ given by a primitive form $f_0 \in S_{k_0}(C,\psi_0)$. We have a modular form H with $H \equiv 1 \mod \mathfrak{p}$ of weight 1 of level p with coefficients in $\mathbb{Z}_p$ and character ω^{-1} for the Teichimüller character ω modulo p. Then $fH^n \equiv f \mod p$, and $i : f \mapsto fH^n$ gives a q-expansion preserving $\mathbb{F}$-linear map $S_{k_0}(Cp,\psi_0)_{/\mathbb{F}} \hookrightarrow S_k(Cp,\psi_k)_{/\mathbb{F}}$ $(k = k_0 + n, \psi_k = \psi_0\omega^{-n})$. Note that $\overline{\chi} = (\nu_p^{k_0-1}\psi_0 \mod \mathfrak{p}) = (\nu_p^{k-1}\omega^{-n}\psi \mod \mathfrak{p})$, and the action of $T(n)$ on $S_k(Cp,\psi_k)_{/\mathbb{F}}$ is

$$a(m,f|T(n)) = \sum_{d|m,d|n,(d,pC)=1} \overline{\chi}(d)a(\frac{mn}{d^2},f) \text{ (e.g., [MFG, §3.1.7])}$$

and hence i is Hecke equivariant. Thus we have $\mathbb{T}_k$ as a factor of $h_k(Cp,\psi_k)_{/W_{\mathfrak{p}}}$ giving the same $\overline{\rho}$. We then define $\mathbb{T} = \mathbb{T}_{\mathfrak{p}}$ to be the subalgebra of $\prod_{k \geq k_0} \mathbb{T}_k$ topologically generated by $T(n)$ for all n (here $T(n)$ has projection to $T(n)$ in $\mathbb{T}_k$ for all $k \geq k_0$). Thus $\mathbb{T}_{\mathfrak{p}}$ is reduced under minimality.

$\S$**2.15. Big Galois representation.** Consider the product $\rho_{\mathbb{T}_\mathfrak{p}} = \prod_{k \geq k_0} \rho_{\mathbb{T}_k} : G \to \mathrm{GL}_2(\prod_{k \geq k_0} \mathbb{T}_k)$. Then $\mathrm{Tr}(\rho_{\mathbb{T}_\mathfrak{p}}(\mathrm{Frob}_l)) = T(l) \in \mathbb{T}_\mathfrak{p}$ for all primes $l \nmid Cp$. By Chebotarev, $\mathrm{Tr}(\rho_\mathfrak{p})$ has values in $\mathbb{T}$; so, by means of pseudo characters, if $\overline{\rho}$ is irreducible, this representation descent to $\rho_{\mathbb{T}_\mathfrak{p}} : G \to \mathrm{GL}_2(\mathbb{T}) \in \mathcal{D}_\mathfrak{p}(\mathbb{T}_\mathfrak{p})$. Our base ring B is $W_\mathfrak{p}$ but we can descend further to the Witt vector ring $W = W(\mathbb{F})$. Since $\det(\rho_{\mathbb{T}_\mathfrak{p}})$ is a deformation of $\det(\overline{\rho})$, we have a canonical algebra structure $i_{\mathbb{T}_\mathfrak{p}} : \Lambda = W[[G_p^{ab}]] \to \mathbb{T}_\mathfrak{p}$. This is the representation constructed in 1986 in my paper published in Inventiones Math. **85** (1986), in which the representation is constructed only assuming that $a(p, f_0) = \lambda(T(p)) \notin \mathfrak{p}$.

Theorem 2.15: *Suppose that $\rho_{f_0,\mathfrak{p}}$ is minimal satisfying* (ord_p) *with irreducible $\overline{\rho}_{f,\mathfrak{p}}$. Then we have a Galois representation $\rho_{\mathbb{T}_\mathfrak{p}} : G \to \mathrm{GL}_2(\mathbb{T}_\mathfrak{p})$ in $\mathcal{D}(\mathbb{T}_\mathfrak{p})$ such that $\mathrm{Tr}(\mathrm{Frob}_l) = T(l)$ for all primes $l \nmid Cp$.*

§**2.16. Rank theorem**. Define $S_k^{ord}(Cp,\psi)_{/A}$ for $A \in CL_W$ by the maximal subspace (and hence the maximal quotient) on which $U(p)$ is invertible. Since $S_{k_0}(Cp,\psi_0)_{/\mathbb{F}} \hookrightarrow S_k(Cp,\psi_k)_{/\mathbb{F}}$ (by a $T(n)$-equivariant map)), we get

$$\mathrm{rank}_W\, S_k^{ord}(Cp,\psi_k)_{/W} \geq \mathrm{rank}_W\, S_{k_0}^{ord}(Cp,\psi_0)_{/W}.$$

Another result in 1986 is (see [GME, §3.2.4] or [LFE, §7.3]).

Theorem 2.16.1: *We have, for all $k \geq 2$,*

$$r := \mathrm{rank}_W\, S_k^{ord}(Cp,\psi_k)_{/W} = \mathrm{rank}_W\, S_{k_0}^{ord}(Cp,\psi_0)_{/W}.$$

Corollary 2.16.2: *$\mathbb{T}_{\mathfrak{p}}$ is reduced Λ-free of rank equal to r and $\mathbb{T}_{\mathfrak{p}}/(t-\chi(\gamma)) \cong \mathbb{T}_{\chi,\mathfrak{p}}$ ($\chi = \nu_p^{k-1}\psi_k$) for all $k \geq 2$. If Taylor–Wiles condition holds for $\overline{\rho}_{\mathfrak{p}}$, we have $R_{\mathfrak{p}}^{ord} \cong \mathbb{T}_{\mathfrak{p}}$. In particular, if $\mathfrak{p} \nmid \mathrm{Ann}(f)$, $R_{\mathfrak{p}}^{ord} \cong \Lambda$.*

We prove the corollary assuming that $R_{\chi,\mathfrak{p}} = \mathbb{T}_{\chi,\mathfrak{p}}$ for all $k \geq 2$, though the first assertion is valid without having $R = \mathbb{T}$ theorem.

§**2.17. Proof of Corollary.** Since $\rho_{\mathbb{T}_\chi} \in \mathcal{D}(\mathbb{T}_\chi)$, we have the universal map $\pi_\chi : R_\chi \to \mathbb{T}_\chi$ with $\pi(\mathrm{Tr}(\boldsymbol{\rho}_\chi)) = \mathrm{Tr}(\rho_{\mathbb{T}_\chi})$; so, π_χ is onto. Since $R^{ord}_{\mathfrak{p}}/(t-\chi(\gamma)) = R_{\chi,\mathfrak{p}}$ and $\mathbb{T}_{\mathfrak{p}}/(t-\chi(\gamma))$ surjects down to $\mathbb{T}_{\chi,\mathfrak{p}}$, we have the commutative diagram:

$$\begin{array}{ccccc} \Lambda^r & \xrightarrow[\pi_0]{\twoheadrightarrow} & R^{ord}_{\mathfrak{p}} & \xrightarrow[\pi_1]{\twoheadrightarrow} & \mathbb{T}_{\mathfrak{p}} \\ & & {\scriptstyle \mathrm{onto}}\downarrow & & \downarrow{\scriptstyle \mathrm{onto}} \\ & & R^{ord}_{\mathfrak{p}}/(t-\chi(\gamma))R^{ord}_{\mathfrak{p}} & \xrightarrow[\sim]{} & \mathbb{T}_{\chi,\mathfrak{p}}. \end{array}$$

The map π_0 is given as follows: Choose $h_1, \dots, h_r \in R^{ord}_{\mathfrak{p}}$ giving a basis of $\mathbb{T}_{\chi,\mathfrak{p}}$ modulo $(t-\chi(\gamma))$. By Nakayama's lemma, $\pi_0(a_1, \dots, a_r) = \sum_i a_i h_i$ is an onto Λ-linear map. Put $\pi = \pi_1 \circ \pi_0$. Thus $\mathrm{Ker}(\pi) \subset \bigcap_\chi (t-\chi(\gamma))\Lambda^r = 0$; so, $\Lambda^r \cong R^{ord} \cong \mathbb{T}_{\mathfrak{p}}$. The last assertion follows from Theorem 2.7. □

§2.18. **Presentation theorem.**

Theorem 2.18: *Assume* $R^{ord}_{\mathfrak{p}} \cong \mathbb{T}_{\mathfrak{p}}$ *and* $\mathbb{T}_{\chi,\mathfrak{p}} \cong \frac{W[[\overline{T}_1,\dots,\overline{T}_r]]}{(\overline{S}_1,\dots,\overline{S}_r)}$ *for* $r = \dim_{\mathbb{F}} t_{\mathbb{T}_{\chi,\mathfrak{p}}/W}$ *for one* $\chi = \nu_p^{k-1}\psi_k$ *($k \geq 2$). Then we have* $R^{ord}_{\mathfrak{p}} \cong \Lambda[[T_1,\dots,T_r]]/(S_1,\dots,S_r)$ *with* $T_j \mod (t-\chi(\gamma)) = \overline{T}_j$ *and* $S_j \mod (t-\chi(\gamma)) = \overline{S}_j$*.*

Proof. We write $\bar{t}_i$ for the image of $\overline{T}_i$ in $\mathbb{T}_\chi$. As remarked in §1.21 $t^*_{\mathbb{T}_\chi/W} = \Omega_{\mathbb{T}_\chi/W} \otimes_{\mathbb{T}_\chi} \mathbb{F} = \Omega_{\mathbb{T}_{\mathfrak{p}}/\Lambda} \otimes_{\mathbb{T}_{\mathfrak{p}}} \mathbb{F} = t^*_{\mathbb{T}_{\mathfrak{p}}/\Lambda}$. Any lifts $\{t_i\}_i$ of $\{\bar{t}_i\}_i$ give rise to a basis of $t^*_{\mathbb{T}_{\mathfrak{p}}/\Lambda}$. Then we have a surjective CL_Λ-morphism $\pi : \Lambda[[T_1,\dots,T_r]] \twoheadrightarrow \mathbb{T}_{\mathfrak{p}}$ with $T_i \mapsto t_i$. Then $\mathrm{Ker}(\pi)/(t-\chi(\gamma))\,\mathrm{Ker}(\pi)$ is generated by $\overline{S}_1,\dots,\overline{S}_r$, and we lift $\overline{S}_i$ to $S_i \in \mathrm{Ker}(\pi)$. So $\mathrm{Ker}(\pi) \otimes_{\Lambda[[T_1,\dots,T_r]]} \mathbb{F}$ is generated by the image of $\overline{S}_i$ as $\mathbb{F}$-vector space; so, by Nakayama's lemma, we have $\mathrm{Ker}(\pi) = (S_1,\dots,S_r)$ as desired. □

§**2.19. Deformation functor over Λ and R^{ord}.** Let $\boldsymbol{\kappa} := \det(\boldsymbol{\rho}^{ord}) : G \to \Lambda^\times$. Then $(\Lambda, \boldsymbol{\kappa})$ represents

$$A \mapsto \{\xi : G \to A^\times | \xi \bmod \mathfrak{m}_A = \det(\overline{\rho})\}.$$

Consider a new deformation functor $\mathcal{D}_{\boldsymbol{\kappa}} : CL_{/\Lambda} \to SETS$:

$$\mathcal{D}_{\boldsymbol{\kappa}}(A) = \{\rho \in \mathcal{D}(A) | \det(\rho) = i_A \circ \boldsymbol{\kappa}\}/\Gamma(\mathfrak{m}_A),$$

slightly different from the one $\mathcal{D}$, where writing $i_A : \Lambda \to A$ for Λ-algebra structure of A. $\mathcal{D}_{\boldsymbol{\kappa}}$ is again represented by $(R^{ord}, \boldsymbol{\rho}^{ord})$ regarding R^{ord} as a Λ-algebra by the CL_W-morphism induced by $\det(\boldsymbol{\rho}) : G \to R^{ord\times}$. Indeed, if $\rho \in \mathcal{D}_{\boldsymbol{\kappa}}(A)$, we have $i_A \circ \boldsymbol{\kappa} = \det(\rho)$. Regarding $\rho \in \mathcal{D}(A)$, we have a unique CL_W-morphism $R^{ord} \xrightarrow{\phi} A$ with $\phi \circ \boldsymbol{\rho} \sim \rho$. Taking determinant, we get $\phi \circ \boldsymbol{\kappa} = \det(\rho)$ showing that ϕ is compatible with $i_{R^{ord}}$ and i_A; so, it is a CL_Λ-morphism, showing $\mathrm{Hom}_\Lambda(R, A) \cong \mathcal{D}_{\boldsymbol{\kappa}}(A)$ by $\phi \leftrightarrow \rho$. Thus by §1.27, $\boxed{\mathrm{Sel}(Ad(\rho))^\vee \cong \Omega_{R^{ord}/\Lambda} \otimes_{R^{ord},\phi} A}$.

§2.20. Fitting ideals.

Let $A \in CL_B$. Let M be an A-module with presentation:

$$A^r \xrightarrow{L} A^s \rightarrow M \rightarrow 0$$

for a matrix L in $M_{s,r}(A)$. If $r \geq s$, the A-ideal $\mathrm{Fitt}_A(M)$ generated by $s \times s$-minors of L is called the Fitting ideal of M, which is independent of the choice of the presentation (and the choice of matrix form of L). If $r < s$, we put $\mathrm{Fitt}_A(M) = 0$.

By definition, $\mathrm{Fitt}_B(M \otimes_{A,\phi} B) = B \cdot \phi(\mathrm{Fitt}_A(M))$ for a CL_B-morphism $\phi : A \rightarrow B$. If $r = s$, $\mathrm{Fitt}_A(M)$ is principal.

See Appendix of Mazur–Wiles paper: Inventiones **76** (1984), 179–330 for a summary of the theory of Fitting ideal. Eisenbud's book: "Commutative algebra, with a view toward algebraic geometry" GTM **150**, 1995, Springer has more details in §20.

§2.21. Examples of Fitting ideals and remarks.

• If $A = B = W$ with $\mathfrak{m}_W = (\varpi)$, $M \cong W/\varpi^{e_1}W \oplus \cdots \oplus W/\varpi^{e_r}W$. Choose $L = \mathrm{diag}[\varpi^{e_1}, \ldots, \varpi^{e_r}]$. If $W = \mathbb{Z}_p$, $\mathrm{Fitt}_{\mathbb{Z}_p} = (|M|)$ and,

$$\mathrm{Fitt}_W(M) = (\prod_i \varpi^{e_i}) = (||M||_p^{-\mathrm{rank}_{\mathbb{Z}_p} W}) \text{ in general.}$$

• If $B = A = \Lambda$ and M is a Λ-torsion module, we have a Λ-linear morphism $i : M \to \Lambda/(f_1) \oplus \cdots \oplus \Lambda/(f_r)$ for $f_j \in \mathfrak{m}_\Lambda$ with finite kernel and finite cokernel. Then we define $\mathrm{char}_\Lambda(M) := (\prod_i f_i)$ (the characteristic ideal with characteristic power series $\prod_i f_i$).

• It is known that $\mathrm{Fitt}_\Lambda(M) = \mathrm{char}_\Lambda(M)$ if $\mathrm{Fitt}_\Lambda(M)$ is principal and $\bigcap_P \mathrm{Fitt}_\Lambda(M)_P = \mathrm{char}_\Lambda(M)$, where P runs over all principal non-zero prime ideals of Λ (i.e., height 1 prime ideal). So for any normal noetherian domain A, we define

$$\mathrm{char}_A(M) := \bigcap_{P:\text{height } 1} \mathrm{Fitt}_A(M)_P.$$

• If we have a good p-adic L-function L_ρ of a Galois representation $\rho : G \to \mathrm{GL}_n(A)$, one expects $\mathrm{char}_A(\mathrm{Sel}(\rho)^\vee) = (L_\rho)$ (Main conjecture).

§**2.22. Tate's theorem** [MFG, §5.3.4].

Theorem 2.22.1: *Suppose B is a domain. Let $A \in CL_B$ be a reduced B-algebra free of finite rank over B. If $A \cong \frac{B[[T_1,\dots,T_r]]}{(S_1,\dots,S_r)}$, then for any B-algebra homomorphism $\lambda : A \to B$,*

$$\mathrm{Fitt}_B(C_1(\lambda; B)) = \mathrm{Fitt}_B(C_0(\lambda, B)),\ C_0(\lambda; B) = B/\,\mathrm{Fitt}_B(C_0(\lambda; B)).$$

We say a Hecke eigenform $f \in S_k(Cp, \psi)_{/W_{\mathfrak{p}}}$ belongs to $\mathbb{T}_{\mathfrak{p}}$ if $\lambda_f : h_k(Cp, \psi)_{/W_{\mathfrak{p}}} \to W_{\mathfrak{p}}$ given by $g|T(n) = \lambda_g(T(n))g$ factors through $\mathbb{T}_{k'}$. For an irreducible component Spec($\mathbb{I}$) $\subset$ Spec($\mathbb{T}_{\mathfrak{p}}$), if λ_g factors through $\mathbb{I}$ is called "belonging to $\mathbb{I}$". The set of all g belonging to $\mathbb{I}$ is called the p-adic analytic family of Hecke eigenform of $\mathbb{I}$ (or the Hida family of $\mathbb{I}$).

Corollary 2.22.2: *Assume that $\overline{\rho}_{\mathfrak{p}}$ is minimal satisfying* (ord$_p$)*. Let $B = W_{\mathfrak{p}}$ or $\Lambda = W_{\mathfrak{p}}[[T]]$, and write $\mathbb{T}_B = \mathbb{T}_{\mathfrak{p}}$ if $B = \Lambda$ and $\mathbb{T}_B = \mathbb{T}_k$ if $B = W_{\mathfrak{p}}$ for $k \geq 2$. Then* $\mathrm{Fitt}_{\mathbb{T}_B}(\Omega_{\mathbb{T}_B/B})$ *is a principal ideal generated by a non-zero divisor $L_B \in \mathbb{T}_B$.*

§2.23. Proof of Corollary.

Recall the second fundamental sequence from [MFG, §5.2.3] for a surjective morphism $\pi : A \twoheadrightarrow C$ with $J := \mathrm{Ker}(\pi)$:

$$J/J^2 \xrightarrow{j \mapsto dj} \Omega_{A/B} \otimes_A C \xrightarrow{da \otimes 1 \mapsto d\pi(a)} \Omega_{C/B} \to 0.$$

Applying this to $A = B[[T_1, \ldots, T_r]]$ with $C = \mathbb{T}_B$ and $J = (S_1, \ldots, S_r)$, we get the following diagram withe exact rows:

$$\begin{array}{ccccccc} J/J^2 & \longrightarrow & \Omega_{B[[T_1,\ldots,T_r]]/B} \otimes_A \mathbb{T}_B & \longrightarrow & \Omega_{\mathbb{T}_B/B} & \longrightarrow & 0 \\ \text{onto}\uparrow S_i \mapsto S_i \bmod J^2 & & \wr\uparrow & & \uparrow\Vert & & \\ \bigoplus_i \mathbb{T}_B S_i & \xrightarrow[d]{} & \bigoplus_j \mathbb{T}_B dT_r & \longrightarrow & \Omega_{\mathbb{T}_B/B} & \longrightarrow & 0. \end{array}$$

Thus $\mathrm{Fitt}_{\mathbb{T}_B}(\Omega_{\mathbb{T}_B/B}) = (L_B)$ for $L_B := \det(d) \in \mathbb{T}_B$. Since $\Omega_{\mathbb{T}_B/B}$ is a torsion B-module, L_B is a non-zero divisor by §2.4 (d5). □

§2.24. Algebraic p-adic L.

We call $P \in \mathrm{Spec}(\mathbb{T}_{\mathfrak{p}})(W_{\mathfrak{p}})$ *arithmetic* (resp. *classical*) if $\lambda = \lambda_P : \mathbb{T}_{\mathfrak{p}} \twoheadrightarrow W_{\mathfrak{p}}$ with $P = \mathrm{Ker}(\lambda_P)$ factors through $\mathbb{T}_k$ for $k \geq 2$ (resp. associated to a Hecke eigenform). Any arithmetic point P is classical. The cusp form associated to a classical point P is written as $f_P = \sum_{n=1}^{\infty} \lambda_P(T(n))q^n$ and $\rho_P := \rho_{f_P,\mathfrak{p}}$.

Theorem 2.24: *Suppose $R^{ord} \cong \mathbb{T}_{\mathfrak{p}}$ and that $\mathbb{T}_{\mathfrak{p}}$ is a local complete intersection relative to Λ. Then for arithmetic $P \in$ $\mathrm{Spec}(\mathbb{T}_{\mathfrak{p}})$, we have $L_\Lambda(P) := \lambda_P(L_\Lambda)$ satisfies $|L_\Lambda(P)|_p^{-\mathrm{rank}_{\mathbb{Z}_p} W_{\mathfrak{p}}} =$ $|\mathrm{Sel}(Ad(\rho_P)|$ or equivalently*

$$L_\Lambda(P) = |\mathrm{Sel}(Ad(\rho_P))|$$

up to units in $W_{\mathfrak{p}}$.

Therefore it is natural to call $L_\Lambda : \mathrm{Spec}(\mathbb{T}_{\mathfrak{p}}) \to \overline{\mathbb{Q}}_p$ the algebraic adjoint p-adic L-function.

§2.25. Proof.
As we have seen in §2.19, by $R^{ord} \cong \mathbb{T}_{\mathfrak{p}}$,

$$\mathsf{Sel}(Ad(\rho_A))^\vee \cong \Omega_{\mathbb{T}_{\mathfrak{p}}/\Lambda} \otimes_{R^{ord},\varphi} A$$

for $\varphi : R^{ord} \to A$ with $\rho_A \sim \varphi \circ \boldsymbol{\rho}^{ord}$. Since as remarked in §2.20,

$$\mathsf{Fitt}_A(\Omega_{\mathbb{T}_{\mathfrak{p}}} \otimes_{\mathbb{T}_{\mathfrak{p}}} A) = A \cdot \varphi(\mathsf{Fitt}_{\mathbb{T}_{\mathfrak{p}}}(\Omega_{\mathbb{T}_{\mathfrak{p}}/\Lambda}))$$

if $\rho_A = \rho_P$ for arithmetic P, taking $A = \mathbb{T}_{\mathfrak{p}}/P = W_{\mathfrak{p}}$, we have

$$L_\Lambda(P) = (L_\Lambda \bmod P)$$

is the generator of $\mathsf{Fitt}_{W_{\mathfrak{p}}}(\Omega_{\mathbb{T}_{\mathfrak{p}}/\Lambda} \otimes_{\mathbb{T}_{\mathfrak{p}}} W_{\mathfrak{p}}) = \mathsf{Fitt}_{W_{\mathfrak{p}}}(\mathsf{Sel}(Ad(\rho_P))$. Thus again by the computation in §2.20 of Fitting ideal for a module over $W = W_{\mathfrak{p}}$, we get

$$L_\Lambda(P) = |\mathsf{Sel}(Ad(\rho_P))|$$

up to units. $\square$

§2.26. p-adic analytic L.

It is known [MFG, §5.3] for the two periods $\Omega_f^{\pm} \in \mathbb{C}^{\times}$:

Theorem 2.26: *Suppose $\mathbb{T}_{\mathfrak{p}}$ is local complete intersection over Λ. Then we have for all arithmetic $P \in \mathrm{Spec}(\mathbb{T}_{\mathfrak{p}})(\overline{\mathbb{Q}}_p)$*

$$|C_0(\lambda_P; W_{\mathfrak{p}})| = \left| \frac{L(1, Ad(f_P))}{\Omega_{f_P}^{+}\Omega_{f_P}^{-}} \right|_p^{-\mathrm{rank}_{\mathbb{Z}_p} W_{\mathfrak{p}}}.$$

$L(s, Ad(f)) = \prod_l \det(1 - \mathrm{Frob}\rho_{f,\mathfrak{q}}(\mathrm{Frob}_l)|_{Ad(\rho_{f,\mathfrak{q}})^{I_l}} l^{-s})^{-1}$. Here we choose a prime ideal $\mathfrak{q}$ of $\mathbb{Z}[f]$ prime to l.

If $f|T(l) = \lambda(T(l))f$ ($l \nmid Cp$), for two roots α, β of $X^2 - \lambda(T(l))X + \chi(l) = 0$, we have

$\det(1 - \mathrm{Frob}\rho_{f,\mathfrak{q}}(\mathrm{Frob}_l)l^{-s}) = (1 - \alpha\beta^{-1}l^{-s})(1 - l^{-s})(1 - \beta\alpha^{-1}l^{-s})$.

§2.27. **Adjoint class number formula.** By the Ramanujan–Pettersson conjecture (proven by Deligne), $|\alpha| = |\beta| = \sqrt{p^{k-1}}$; so, $L(s, Ad(f))$ converges absolutely if $\mathrm{Re}(s) \geq 1$, and by Shimura, it has analytic continuation to the whole complex plane. Theorem 2.26 combined with Tate's theorem implies

Corollary 2.27: *Suppose $R^{ord} \cong \mathbb{T}_{\mathfrak{p}}$ and that $\mathbb{T}_{\mathfrak{p}}$ is local complete intersection over Λ. Then a generator L_Λ of $\mathrm{Fitt}_{\mathbb{T}_{\mathfrak{p}}}(\Omega_{\mathbb{T}_{\mathfrak{p}}/\Lambda})$ satisfies $L_\Lambda(P) = |\mathrm{Sel}(Ad(\rho_P))| = \left|\frac{L(1,Ad(f_P))}{\Omega^+_{f_P}\Omega^-_{f_P}}\right|_p^{-\mathrm{rank}_{\mathbb{Z}_p} W_{\mathfrak{p}}}$ up to unit for all arithmetic points $P \in \mathrm{Spec}(\mathbb{T}_{\mathfrak{p}})$.*

For example, if f is associated to an elliptic curve $E_{/\mathbb{Q}}$, then choosing a generator $\gamma_\pm \in H_0(E(\mathbb{C}), \mathbb{Z})^\pm$ for the $\pm$-eigenspace of complex conjugation, $\Omega^\pm_f = \int_{\gamma_\pm} 2\pi i f dz = \int_{\gamma_\pm} \frac{df}{dq} dq$. We hope to come back to the analytic theory towards the end of this course.