We define $\text{Sel}(\text{Ad}(\rho_A))$ for ordinary deformations $\rho_A \in \mathcal{D}_\chi(A)$ of an **absolutely irreducible** 2-dimensional minimal Galois representation $\bar{\rho}$ and show that $\text{Sel}(\text{Ad}(\bar{\rho})) = t_{R/B}$ and $\text{Sel}(\text{Ad}(\rho_A))^\vee \cong \Omega_{R/B} \otimes_{R,\varphi} A$, where $\varphi : R \to A$ with $\varphi \circ \rho \sim \rho_A$ for the universal minimal ordinary Galois representation $\rho : G \to \text{GL}_2(R)$ of $\bar{\rho}$. Here the deformation functors $\mathcal{D}, \mathcal{D}_\chi : \mathcal{C} \to \text{SETS}$ are defined in §0.22.

As before, we write $I_l$ for the inertia group of the $l$-decomposition subgroup $D_l \subset G$. We write $S$ for the set of ramified primes $l \neq p$ of $\bar{\rho}$ such that $\bar{\rho}|_{I_l} \cong \bar{\varepsilon}_l \oplus \bar{\delta}_l$. We set $\mathbb{F}[\varepsilon] := \mathbb{F}[X]/(X^2)$ (dual numbers) with $\varepsilon \leftrightarrow X \mod (X^2)$. 
§1.1. \( p \)-Ordinarity condition

Fix \( \bar{\rho} : G \to \text{GL}_2(\mathbb{F}) \) with \( \bar{\rho} = \rho_A|_{D_p} \cong \begin{pmatrix} \overline{\epsilon} & * \\ 0 & \overline{\delta} \end{pmatrix} \) and \( \overline{\epsilon} \neq \overline{\delta} \). Let \( \rho_A : G \to \text{GL}_2(A) \ (A \in \mathcal{C}) \) be a deformation of \( \bar{\rho} : G \to \text{GL}_2(\mathbb{F}) \) acting on \( V(\rho_A) \). We say \( \rho \) is \( p \)-ordinary if
\[
(\text{ord}_p) \rho_A|_{D_p} \cong \begin{pmatrix} \epsilon_A & * \\ 0 & \delta_A \end{pmatrix}
\]
for two characters \( \epsilon_A, \delta_A : D_p \to A^\times \) distinct modulo \( m_A \) with \( \delta_A \) unramified with \( \delta_A \mod m_A = \overline{\delta} \) (this is a requirement called \( p \)-distinguishedness).

Since twisting by a character \( \xi : G \to B^\times \) induces isomorphism between the functors deforming \( \bar{\rho} \) and \( \bar{\rho} \otimes \xi \), we may assume a similar condition for \( l \in S \ (l \neq p) \):
\[
(\text{ord}_l) \rho|_{I_l} \cong \begin{pmatrix} \epsilon_{l,A} & 0 \\ 0 & 1 \end{pmatrix}
\]
with \( \epsilon_{l,A} \neq 1 \).

We can fix a character \( \chi : G \to B^\times \), we consider
\( (\text{det}) \det \rho = \iota_A \circ \chi \) for the \( B \)-algebra structure \( \iota_A : B \to A \).

The fixed determinant functor is denoted by \( \mathcal{D}_\chi : \mathcal{C} \to \text{SETS} \).
§1.2. **Deformation functor.**

We consider the following functors for a fixed absolutely irreducible representation $\overline{\rho} : G \to \text{GL}_2(\mathbb{F})$ satisfying $(\text{ord}_p)$ and $(\text{ord}_l)$. Recall $D^\emptyset, D, D_\chi : C \to \text{SETS}$ given by

$$
D^\emptyset(A) := \{\rho_A : G \to \text{GL}_2(A) | \rho_A \mod m_A = \overline{\rho}\} / \Gamma(m_A),
$$

$$
D(A) = \{\rho_A \in D^\emptyset(A) | (\text{min}), (\text{ord}_p) \text{ and } (\text{ord}_l)\},
$$

$$
D_\chi(A) = \{\rho_A \in D(A) | \text{det } \rho = \iota_A \circ \chi\}.
$$

Then

**Theorem 1** (B. Mazur). *There exists universal couples $(R, \rho)$, $(R^{\text{ord}}, \rho^{\text{ord}})$ and $(R_\chi, \rho_\chi)$ representing $D^\emptyset$, $D$ and $D_\chi$, respectively, so that $D(A) \cong \text{Hom}_C(R^{\text{ord}}, A)$ by $\rho \mapsto \varphi$ with $\varphi \circ \rho^{\text{ord}} \sim \rho$ (resp. $D_\chi(A) \cong \text{Hom}_C(R_\chi, A)$ by $\rho \mapsto \varphi$ with $\varphi \circ \rho_\chi \sim \rho$).

We admit this theorem (see [MFG, §2.3] or Mazur’s paper quoted there).
§1.3. Fiber products.
Let $C = C, SETS$. For arrows $\phi' : S' \to S$ and $\phi'' : S'' \to S$ in $C$,
$$S' \times_S S'' = \{(a', a'') \in S' \times S'' | \phi'(a') = \phi''(a'')\}$$
gives the fiber product of $S'$ and $S''$ over $S$ in $C$. So
$$\text{Hom}_C(X, S' \times_S S'') = \text{Hom}_C(X, S') \times_{\text{Hom}_C(X, S)} \text{Hom}_C(X, S'')$$
for any $X \in C$. Let $\mathcal{F} : C \to SETS$ be a covariant functor. We assume
$$|\mathcal{F}(\mathcal{F})| = 1$$
and
$$\mathcal{F}(\mathcal{F}[\varepsilon] \times_{\mathcal{F}} \mathcal{F}[\varepsilon]) = \mathcal{F}(\mathcal{F}[\varepsilon]) \times_{\mathcal{F}(\mathcal{F})} \mathcal{F}(\mathcal{F}[\varepsilon])$$
by two projections.

It is easy to see $\mathcal{F} \in \{ \mathcal{D}^0, \mathcal{D}, \mathcal{D}_\chi \}$ satisfies this condition. Indeed, noting that $\mathcal{F}[\varepsilon] \times_{\mathcal{F}} \mathcal{F}[\varepsilon] \cong \mathcal{F}[\varepsilon'] \times_{\mathcal{F}} \mathcal{F}[\varepsilon''] \cong \mathcal{F}[\varepsilon', \varepsilon'']$, if $\rho' \in \mathcal{F}(\mathcal{F}[\varepsilon'])$ and $\rho'' \in \mathcal{F}(\mathcal{F}[\varepsilon''])$, we have $\rho' \times \rho''$ has values in $\text{GL}_2(\mathcal{F}[\varepsilon', \varepsilon''])$ is an element in $\mathcal{F}(\mathcal{F}[\varepsilon'] \times_{\mathcal{F}} \mathcal{F}[\varepsilon''])$. 
§1.4. Tangent space of deformation functors.

For $A \in \mathcal{C}$ and an $A$-module $X$, suppose

$$|\mathcal{F}(A)| = 1 \text{ and } \mathcal{F}(A[X] \times_A A[X]) = \mathcal{F}(A[X]) \times_{\mathcal{F}(A)} \mathcal{F}(A[X]).$$

Note $A[X] \times_A A[X] = A[X \oplus X]$. The addition on $X$ and $A$-linear map $\alpha : X \to X$ induces in the same way $\mathcal{C}$-morphisms

$\cdot \cdot \cdot : A[X \oplus X] \to A[X]$ by $a + (x \oplus y) \mapsto a + x + y$ and $\alpha_* : A[X] \to A[X]$ by $a + x \mapsto a + \alpha(x)$. Thus we have by functoriality, the “addition”

$$\cdot \cdot \cdot : \mathcal{F}(A[X]) \times_{\mathcal{F}(A)} \mathcal{F}(A[X]) = \mathcal{F}(A[X \oplus X]) \xrightarrow{\mathcal{F}(\cdot \cdot \cdot)} \mathcal{F}(A[X])$$

and $\alpha$-action

$$\alpha : \mathcal{F}(A[X]) \xrightarrow{\mathcal{F}(\alpha_*)} \mathcal{F}(A[X]).$$

With $0 = \text{Im}(\mathcal{F}(A) \to \mathcal{F}(A[X]))$ for the inclusion $A \hookrightarrow A[X]$, this makes $\mathcal{F}(A[X])$ as an $A$-module; so, $\mathcal{F}(\mathbb{F}[\varepsilon])$ is an $\mathbb{F}$-vector space (called the tangent space of $\mathcal{F}$).
§1.5. Cotangent spaces of local rings

Suppose that $B$ is noetherian and pick $R \in CL_B$.

Lemma 1. The ring $R$ is noetherian if and only if $t*_{R/W} = m_R/(m_R^2 + m_B)$ is a finite dimensional vector space over $\mathbb{F}$.

The space $t*_{R/B}$ is called the co-tangent space of $R$ at $m_R = (\varpi) \in \text{Spec}(R)$ over $\text{Spec}(B)$. If $m_B = (x_1, \ldots, x_r)$, then $m_B^n/m_B^{n+1}$ is generated by degree $n$ monomial of $x_j$; so, $B/m_B^n$ is generated by degree $\leq n$ polynomial of $x_j$. Thus for $W = W(\mathbb{F})$, $W[X_1, \ldots, X_r]$ has dense image in $B$ by sending $x_j$ to $X_j$, and hence $W[[X_1, \ldots, X_r]] \rightarrow B$.

Since we have an exact sequence: $m_B/m_B^2 \rightarrow m_R/m_R^2 \rightarrow t*_{R/W}$, we conclude in the same way that $W[[X_1, \ldots, X_r, X_{r+1}, \ldots, X_{r+s}]]$ surjects onto $R$ sending $X_i$ with $i > r$ to generators of $t*_{R/B}$. Thus the number of generators over $B$ of $R$ is $\dim_{\mathbb{F}} t*_{R/B}$. 
§1.6. Adjoint Galois modules

Let $M_2(A)$ be the space of $2 \times 2$ matrices with coefficients in $A$. We let $G$ acts on $M_2(A)$ by $gv = \rho_A(g)v\rho_A(g)^{-1}$. This action is called the **adjoint** action of $G$, and this $G$–module will be written as $ad(\rho_A)$.

Write $Z$ for the center of $M_2(A)$ (scalar matrices) and define $\mathfrak{sl}_2(A) = \{X \in M_2(A)|\text{Tr}(X) = 0\}$. Since $\text{Tr}(aXa^{-1}) = \text{Tr}(X)$, $\mathfrak{sl}_2(A)$ is stable under the adjoint action. This Galois module will be written as $Ad(\rho_A)$.

Since $p > 2$, $X \mapsto \frac{1}{2}\text{Tr}(X) \oplus (X - \frac{1}{2}\text{Tr}(X))$ gives rise to $M_2(A) = Z \oplus \mathfrak{sl}_2(A)$ stable under the adjoint action.

So we have $ad(\rho_A) = 1 \oplus Ad(\rho_A)$, where 1 is the trivial representation.
§1.7. Tangent space as cohomology

Lemma 2. Let $(R, \rho)$ be the universal couple representing $D^0$ over $CL_W$. Then

$$t_{R/W} \coloneqq \text{Hom}_{F}(t^*_R/W, F) \cong H^1(G, \text{ad}(\rho)),$$

where $H^1(G, \text{ad}(\rho))$ is the continuous first cohomology group of $G$ with coefficients in the discrete $G$–module $V(\text{ad}(\rho))$.

Proof, Step 1, dual number.

We claim: $\text{Hom}_{CL_W}(R, F[\varepsilon]) \cong t_{R/W}$. Construction of the map.

Start with a $W$-algebra homomorphism $\phi : R \to F[\varepsilon]$. Write

$$\phi(r) = \phi_0(r) + \phi_\varepsilon(r)\varepsilon \quad \text{with} \quad \phi_0(r), \phi_\varepsilon(r) \in F.$$

Then the map is $\phi \mapsto \ell_\phi = \phi_\varepsilon|_{m_R}$. 
§1.8. Step. 2, Well defined-ness of $\ell_\phi$

From $\phi(ab) = \phi(a)\phi(b)$, we get

$$\phi_0(ab) = \phi_0(a)\phi_0(b)$$

and $\phi_\varepsilon(ab) = \phi_0(a)\phi_\varepsilon(b) + \phi_0(b)\phi_\varepsilon(a)$. Thus $\phi_\varepsilon \in \text{Der}_W(R, \mathbb{F}) \cong \text{Hom}_\mathbb{F}(\Omega_{R/W} \otimes_R \mathbb{F}, \mathbb{F})$. Since for any derivation $\delta \in \text{Der}_W(R, \mathbb{F})$, $\phi' = \phi_0 + \delta \varepsilon \in \text{Hom}_{CLW}(R, \mathbb{F}[\varepsilon]),$

$$\text{Hom}_R(\Omega_{R/W} \otimes_R \mathbb{F}, \mathbb{F}) \cong \text{Hom}_R(\Omega_{R/W}, \mathbb{F}) \cong \text{Der}_W(R, \mathbb{F}[\varepsilon]) \cong \text{Hom}_{CLW}(R, \mathbb{F}[\varepsilon]).$$

Note $\text{Ker}(\phi_0) = m_R$ because $R$ is local. Since $\phi$ is $W$–linear, $\phi_0(a) = \overline{a} = a \mod m_R$. Thus $\phi$ kills $m_R^2$ and takes $m_R$ $W$–linearly into $m_{\mathbb{F}[\varepsilon]} = \mathbb{F}\varepsilon$; so, $\ell_\phi : t_R^* := m_R/m_R^2 \rightarrow \mathbb{F}$. For $r \in W$, $\overline{r} = r\phi(1) = \phi(r) = \overline{r} + \phi_\varepsilon(r)\varepsilon$, and hence $\phi_\varepsilon$ kills $W$; so, $\ell_\phi \in t_{R/W}$. 
§1.9. Step. 3, $\phi \mapsto \ell_\phi$ is an injection.

Since $R$ shares its residue field $\mathbb{F}$ with $W$, any element $a \in R$ can be written as $a = r + x$ with $r \in W$ and $x \in m_R$.

Thus $\phi$ is completely determined by the restriction $\ell_\phi$ of $\phi_\varepsilon$ to $m_R$, which factors through $t_{R/W}^*$.

Thus $\phi \mapsto \ell_\phi$ induces an injective linear map $\ell : \text{Hom}_{W-\text{alg}}(R, \mathbb{F}[\varepsilon]) \hookrightarrow \text{Hom}_{\mathbb{F}}(t_{R/W}^*, \mathbb{F})$.

Note $R/(m_R^2 + m_W) = \mathbb{F} \oplus t_{R/W}^* = \mathbb{F}[t_{R/W}^*]$ with the projection $\pi : R \to t_{R/W}^*$ to the direct summand $t_{R/W}^*$. Indeed, writing $\bar{r} = (r \mod m_R)$, for the inclusion $\iota : \mathbb{F} = W/m_W \hookrightarrow R/(m_R^2 + m_W)$, $\pi(r) = r - \iota(\bar{r})$. 
1.10. Step 4, $\phi \mapsto \ell_\phi$ is a surjection.

For any $\ell \in \text{Hom}_\mathbb{F}(t^*_R/W, \mathbb{F})$, we extends $\ell$ to $R$ by putting $\ell(r) = \ell(\pi(r))$. Then we define $\phi : R \to \mathbb{F}[\varepsilon]$ by $\phi(r) = \bar{r} + \ell(\pi(r))\varepsilon$. Since $\varepsilon^2 = 0$ and $\pi(r)\pi(s) = 0$ in $\mathbb{F}[t^*_R/W]$, we have

$$rs = (\bar{r} + \pi(r))(\bar{s} + \pi(s)) = \bar{r}s + \bar{s}\pi(r) + \bar{r}\pi(s)$$

$$\xrightarrow{\phi} \bar{r}s + \bar{s}\ell(\pi(r))\varepsilon + \bar{r}\ell(\pi(s))\varepsilon = \phi(r)\phi(s)$$

is an $W$–algebra homomorphism. In particular, $\ell(\phi) = \ell$, and hence $\ell$ is surjective.

By $\text{Hom}_R(\Omega_{R/W} \otimes_R \mathbb{F}, \mathbb{F}) \cong \text{Hom}_{CLW}(R, \mathbb{F}[\varepsilon])$, we have

$$\text{Hom}_R(\Omega_{R/W} \otimes_R \mathbb{F}, \mathbb{F}) \cong \text{Hom}_{\mathbb{F}}(t^*_R/W, \mathbb{F});$$

so, if $t^*_R/W$ is finite dimensional, we also get

$$\Omega_{R/W} \otimes_R \mathbb{F} \cong t^*_R/W.$$
§1.11. Step. 5, use of universality.

By the universality, we have

\[ \text{Hom}_{CL_B}(R, \mathbb{F}[\varepsilon]) \cong \{ \rho : G \to GL_2(\mathbb{F}[\varepsilon]) \mid \rho \mod m_{\mathbb{F}[\varepsilon]} = \bar{\rho} \} / \sim. \]

Write \( \rho(g) = \bar{\rho}(g) + u'_{\phi}(g)\varepsilon \) for \( \rho \) corresponding to \( \phi : R \to \mathbb{F}[\varepsilon] \). From the mutiplicativity, we have

\[
\bar{\rho}(gh) + u'_{\phi}(gh)\varepsilon = \rho(gh) = \rho(g)\rho(h) \\
= \bar{\rho}(g)\bar{\rho}(h) + (\bar{\rho}(g)u'_{\phi}(h) + u'_{\phi}(g)\bar{\rho}(h))\varepsilon,
\]

Thus as a function \( u' : G \to M_n(\mathbb{F}) \), we have

\[
u'_{\phi}(gh) = \bar{\rho}(g)u'_{\phi}(h) + u'_{\phi}(g)\bar{\rho}(h). \quad (1)\]
§1.12. Step. 6, Getting 1-cocycle.

Define a map $u_{\rho} = u_{\phi} : G \to ad(\bar{\rho})$ by
$$u_{\phi}(g) = u'_{\phi}(g)\bar{\rho}(g)^{-1}.$$ Then by a simple computation, we have
$$gu_{\phi}(h) = \bar{\rho}(g)u_{\phi}(h)\bar{\rho}(g)^{-1}$$
from the definition of $ad(\bar{\rho})$. Then from the above formula (1), we conclude that
$$u_{\phi}(gh) = gu_{\phi}(h) + u_{\phi}(g).$$
Thus $u_{\phi} : G \to ad(\bar{\rho})$ is a 1–cocycle. Thus we get an $\mathbb{F}$-linear map
$$t_{R/W} \cong \text{Hom}_{CL_W}(R, \mathbb{F}[\varepsilon]) \to H^1(G, ad(\bar{\rho}))$$
by $\ell_{\phi} \mapsto [u_{\phi}]$.
§1.13. Step. 7, End of proof.

By computation, for $x \in ad(\bar{\rho})$

$$\rho \sim \rho' \iff \bar{\rho}(g) + u'(g) \varepsilon = (1 + x\varepsilon)(\bar{\rho}(g) + u'(g) \varepsilon)(1 - x\varepsilon)$$

$$\iff u'(g) = x\bar{\rho}(g) - \bar{\rho}(g)x + u'(g)$$

$$\iff u(\bar{\rho})(g) = (1 - g)x + u'(g).$$

Thus the cohomology classes of $u_\rho$ and $u_{\rho'}$ are equal if and only if $\rho \sim \rho'$. This shows:

$$\text{Hom}_F(t^*_R/W, F) \cong \text{Hom}_{W-alg}(R, F[\varepsilon]) \cong$$

$$\{ \rho : G \to GL_2(F[\varepsilon]) | \rho \mod m_{F[\varepsilon]} = \bar{\rho} \}/ \sim$$

$$\cong H^1(G, ad(\bar{\rho})).$$

In this way, we get a bijection between $\text{Hom}_F(t^*_R/W, F)$ and $H^1(G, ad(\bar{\rho}))$. 
§1.14. Tangent space of rings and deformation functor

Lemma 3. Let $\mathcal{F} = \mathcal{D}^0, \mathcal{D}, \mathcal{D}_\chi$ and $R, R^{ord}$ or $R_\chi$ accordingly. Then $t_{R/B} \cong \mathcal{F}(\mathbb{F}[\varepsilon])$ as $\mathbb{F}$-vector spaces.

Proof. Let $R$ be the universal ring for $\mathcal{D}^0$. We have got a canonical bijection in §1.7:

$$
\mathcal{D}^0(\mathbb{F}[\varepsilon]) \xrightarrow{1-1 \text{ onto}} H^1(G, ad(\bar{\rho})) \xrightarrow{i} t_{R/B}
$$

with a vector space isomorphism $i$. We have constructed a cocycle $u_\rho$ from $\rho \in \mathcal{F}(\mathbb{F}[\varepsilon])$ writing $\rho = \bar{\rho} + u_\rho \bar{\rho} \varepsilon$. Regarding $(\rho, \rho') \in \mathcal{F}(\mathbb{F}[\varepsilon]) \times \mathcal{F}(\mathbb{F}[\varepsilon]) = \mathcal{F}(\mathbb{F}[\varepsilon] \times_{\mathbb{F}} \mathbb{F}[\varepsilon])$, we see that $+(\rho, \rho') = \bar{\rho} + (u_\rho \bar{\rho} + u_{\rho'} \bar{\rho}) \varepsilon \in \mathcal{F}(\mathbb{F}[\varepsilon])$; so, $i_1$ is a homomorphism. Similarly, one can check that it is $\mathbb{F}$-linear. Same for $R^{ord}$ and $R_\chi$. \qed
§1.15. Galois deformation ring is noetherian.
Let $H = \text{Gal}(F(p)(\overline{\rho})/F(\overline{\rho}))$. Note that $H^{ab} = C_{F(\overline{\rho})}(p^\infty) = \lim_{\leftarrow n} Cl_{F(\overline{\rho})}(p^\infty)/Cl_{F(\overline{\rho})}(p^\infty)p^n$ and we have an exact sequence for the integer ring $O$ of $F(\overline{\rho})$:

$$\hat{O}_p^\times \to H^{ab} \to C_{F(\overline{\rho})} \to 1.$$ 

Therefore $H^{ab}$ is a $\mathbb{Z}_p$-module of finite type, which tells us finiteness of $\text{Hom}(H^{ab}, \text{ad}(\overline{\rho}))$. By inflation-restriction sequence,

$$0 \to H^1(F(\overline{\rho})/\mathbb{Q}, \text{ad}(\overline{\rho})) \to H^1(G, \text{ad}(\overline{\rho})) \to \text{Hom}(H^{ab}, \text{ad}(\overline{\rho}))$$

is exact. Since $[F(\overline{\rho}) : \mathbb{Q}] < \infty$ and $|\text{ad}(\overline{\rho})| < \infty$, $H^1(F(\overline{\rho})/\mathbb{Q}, \text{ad}(\overline{\rho}))$ is finite. Thus $H^1(G, \text{ad}(\overline{\rho})) \cong t_{R/W}$ is finite. Then by the lemma in §1.14, $R$ is noetherian. This also tells us that $R^{ord}$ and $R_\chi$ are noetherian.
§1.16 Tangent space with local condition.

We regard $\mathcal{F}(\mathbb{F}[\varepsilon]) \subset H^1(G, ad(\rho))$. We may choose by $(\text{ord}_p)$ a basis (dependent on $l \in S \cup \{p\}$) of $V(\rho)$ for $\rho \in \mathcal{F}(\mathbb{F}[\varepsilon])$ so that $\rho|_{D_p}$ is upper triangular with quotient character $\delta$ congruent to $\bar{\delta}$ modulo $m_A$. Similarly by $(\text{ord}_l)$, we choose the basis so that $\rho|_{I_l} = \varepsilon_l \oplus 1$ in this order.

**Theorem 2.** A 1-cocycle $u$ gives rise to a class in $\mathcal{D}_\chi(\mathbb{F}[\varepsilon])$ if and only if $u|_{D_p}$ is upper triangular, $u|_{I_p}$ is upper nilpotent and $\text{Tr}(u) = 0$ over $G$, where $\bar{\nu} = \nu \mod (\varepsilon)$.

For primes $l \neq p$, $u(I_l) = 0$ as $p \nmid |I_l|$ (minimality). The description of cocycles $u$ is independent of $\chi$; so, the tangent space $t_{R_\chi/B}$ is independent as a cohomology subgroup as long as $\mathbb{F}$ does not change.
§1.17. Proof.

By (det), \( 1 = \text{det}(\rho \bar{\rho}^{-1}) = 1 + u_\rho \epsilon = 1 + \text{Tr}(u_\rho) \epsilon; \) so, (det) \( \Leftrightarrow \) \( \text{Tr}(u) = 0 \) over \( G. \) Thus we \( t_{R_x/B} \subset H^1(G, \text{Ad}(\bar{\rho})). \)

Choose a generator \( w \in V(\epsilon) \) over \( \mathbb{F}[\epsilon]. \) Then \( (w, v) \) is a basis of \( V(\rho) \) over \( \mathbb{F}[\epsilon]. \) Let \( (\bar{w}, \bar{v}) = (w, v) \mod \epsilon \) and identify \( V(\text{ad}(\bar{\rho})) \) with \( M_2(\mathbb{F}) \) with this basis. Then defining \( \bar{\rho} \) by \( (\sigma \bar{w}, \sigma \bar{v}) = (\bar{w}, \bar{v})\bar{\rho}(\sigma), \) for \( \sigma \in D_p, \) we have \( \bar{\rho}(\sigma) = \begin{pmatrix} \bar{\epsilon}(\sigma) & * \\ 0 & \delta(\sigma) \end{pmatrix} \) (upper triangular). If \( \sigma \in I_p, \rho \bar{\rho}^{-1} = 1 + u_\rho \) with lower right corner of \( u_\rho \) has to vanish as \( \delta = 1 \) on \( I_p, \) we have \( u_\rho(\sigma) \in \{(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix})\}. \)

The condition \((\text{ord}_p)\) is equivalent to \( u_\rho \) is of the form \((\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix})\) but by \( \text{Tr}(u_\rho) = 0, \) it has to be upper nilpotent; i.e., \((\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix})\). \( \Box \)
§1.18. Adjoint Selmer group. For $\mathcal{F} = \mathcal{D}$ or $\mathcal{D}_\chi$, we define the local deformation functor $\mathcal{D}_{\chi,p}$ by sending $A$ to

$$\{\rho_A : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \text{GL}_2(A)|\rho_A \mod m_A = \bar{\rho} \text{ and } (\text{ord}_p) \text{ and } (\det)\}.$$  

By the proof of the theorem in §1.16, $\mathcal{D}_{\chi,p}(\mathbb{F}[\varepsilon])$ is the space of cohomology classes in $H^1(D_p, \text{Ad}(\bar{\rho}))$ upper triangular over $D_p$ and upper nilpotent over $I_p$. Define $\text{Ad}(\rho_A)$ by the conjugation action on $\mathfrak{sl}_2(A)$ by $\rho_A$, and put $\text{Ad}(\rho_A)^* := \text{Ad}(\rho_A) \otimes_A A^\vee$ (discrete), writing $A^\vee = \text{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ (Pontryagin dual). Define

$$\text{Sel}(\text{Ad}(\rho_A)) := \ker(H^1(G, \text{Ad}(\rho_A)^*) \to \frac{H^1(D_p, \text{Ad}(\rho_A)^*)}{F_-^1 H^1(D_p, \text{Ad}(\rho_A)^*)}),$$

where $F_-^1 H^1(D_p, \text{Ad}(\rho_A)^*) \subset H^1(D_p, \text{Ad}(\rho_A)^*)$ is made of cohomology classes upper triangular over $D_p$ and upper nilpotent over $I_p$. Then we have $\text{Sel}(\text{Ad}(\bar{\rho})) := t_{R_\chi/B}$. 
§1.19. $R^{ord}$ is an algebra over the Iwasawa algebra

The finite order character $\det(\overline{\rho})$ factors through $\text{Gal}(\mathbb{Q}[\mu_{N_0}]/\mathbb{Q})$ for some positive integer $N_0$. Let $N_0$ be the minimal such integer (called conductor of $\det(\overline{\rho})$). Write $N_0 = Np^\nu$ for $N$ prime to $p$; so, $N$ is the prime to $p$-conductor of $\det(\overline{\rho})$.

If $\rho_A$ is a minimal deformation of $\overline{\rho}$, then $\rho_A(I_l) \cong \overline{\rho}(I_l)$ and hence $\det(\rho_A)(I_l) = \det(\overline{\rho})(I_l)$. Therefore, $\det(\rho^{ord})$ is a minimal deformation of $\det(\overline{\rho})$.

By universality, for the universal character $\kappa : G \to W[[\Gamma]]^\times$, we have a (unique) algebra homomorphism $i = i_{R^{ord}} : W[[\Gamma]] \to R^{ord}$ such that $i_{R^{ord}} \circ \kappa = \det(\rho)$. Therefore

$$R^{ord} \text{ is canonically an algebra over } \Lambda = W[[\Gamma]].$$
§1.20. Reinterpretation of $\mathcal{D}$

Consider the following deformation functor $\mathcal{D}_\kappa : \text{CL}/\Lambda \to \text{SETS}$

$$\mathcal{D}_\kappa(A) = \{ \rho : G \to \text{GL}_2(A) | \rho \mod m_A \cong \bar{\rho}, \rho \text{ satisfies } (\text{ord}_p), (\text{ord}_l) \text{ and } (\text{det}_\Lambda) \}/ \cong,$$

where writing $i_A : \Lambda \to A$ for $\Lambda$-algebra structure of $A$, $$(\text{det}_\Lambda) \det(\rho) = i_A \circ \kappa.$$

Proposition 1. We have $\mathcal{D}_\kappa(A) \cong \text{Hom}_{\text{CL}_\Lambda}(R^{\text{ord}}, A)$ with universal representation $\rho^{\text{ord}} \in \mathcal{D}(R^{\text{ord}})$; so,

$$\text{Sel}(\text{Ad}(\bar{\rho})) := t_{R^{\text{ord}}/\Lambda} = \text{Ker}(H^1(G, \text{Ad}(\bar{\rho})) \to \frac{H^1(D_p, \text{Ad}(\bar{\rho}))}{F^+ H^1(D_p, \text{Ad}(\bar{\rho}))}.$$
§1.21. **Proof.** For any $\rho_A \in \mathcal{D}_\kappa(A)$, regard $\rho_A \in \mathcal{D}(A)$. Then we have $\varphi \in \text{Hom}_C(R^{ord}, A)$ such that $\varphi \circ \rho^{ord} \cong \rho_A$. Thus $\varphi \circ \det(\rho^{ord}) = \det(\rho_A)$. Since $\det(\rho_A) = \iota_A \circ \kappa$ and $\det(\rho^{ord}) = \iota_{R^{ord}} \circ \kappa$, we find $\varphi \circ \iota_{R^{ord}} = \iota_A$, and hence $\varphi \in \text{Hom}_{CL\Lambda}(R^{ord}, A)$. This shows that $R^{ord}$ also represents $\mathcal{D}_\kappa$ over $\Lambda$.

As we already remarked, $\mathcal{D}_\kappa(\mathbb{F}[\varepsilon]) = t_{R^{ord}/\Lambda} = m_{R^{ord}}/m_{R^{ord}}^2 + m_\Lambda$ is independent as a subgroup of $H^1(G, Ad(\overline{\rho}))$; so, we get a new expression of $\text{Sel}(Ad(\overline{\rho}))$. 

By the proof, $\Omega_{R^{ord}/\Lambda} \otimes_{R^{ord}} \mathbb{F} \cong \text{Sel}(Ad(\overline{\rho})) \cong \Omega_{R_\chi/B} \otimes_{R_\chi} \mathbb{F}$, so the smallest number of generators of $\Omega_{R^{ord}/\Lambda}$ as $R^{ord}$-modules and $\Omega_{R_\chi/B}$ as $R_\chi$ modules is equal. In the same way, the number of generators of $R^{ord}$ as $\Lambda$-algebras and $R_\chi$ as $B$-algebras is equal.
§1.22. Recall the compatible choice of $\rho_A$. By $(\text{ord}_l)$ for $l \in S \cup \{p\}$, the universal representation $\rho_\chi$ is equipped with a basis $(v_l, w_l)$ so that the matrix representation with respect this basis satisfies $(\text{ord}_l)$. By universality, each class $c \in D_\chi(A)$ has $\rho$ such that $V(\rho) = V(\rho_\chi) \otimes_{R_\chi, \varphi} A$ for a unique $\varphi \in \text{Hom}_{CLB}(R_\chi, A)$, we can choose a unique $\rho_A \in c$ with a basis $\{(v_l = v_l \otimes 1, w_l = w_l \otimes 1)\}_l$ satisfying $\{(\text{ord}_l): l \in S \cup \{p\}\}$ compatible with specialization. We choose such a specific representative $\rho_A$ for each $c \in D_\chi(A)$.

Start with $\rho_A$ as above. Take a finite $A$-module $X$ and consider the ring $A[X] = A \oplus X$ with $X^2 = 0$. Then $A[X]$ is still $p$-profinite. Pick $\rho \in \mathcal{F}(A[X])$ such that $\rho \mod X \sim \rho_A$. By our choice of representative $\rho$ and $\rho_A$ as above, we may (and do) assume $\rho \mod X = \rho_A$. 
§1.23. **General cocycle construction.** Here we allow $\chi = \kappa$ but if $\chi = \kappa$. Letting $B = W$ if $\chi$ has values in $W^\times$ and $\Lambda$ if $\chi = \kappa$, the functor $\mathcal{F} = D_\chi$ is defined over $CL_B$. Let $\rho_A$ act on $M_2(A)$ and $\mathfrak{sl}_2(A) = \{x \in M_2(A) | \text{Tr}(x) = 0\}$ by conjugation. Write this representation $ad(\rho_A)$ and $Ad(\rho_A)$ as before. Let $ad(X) = ad(\rho_A) \otimes_A X$ and $Ad(X) = Ad(\rho_A) \otimes_A X$ and regard them as $G$-modules by the action on $ad(\rho_A)$ and $Ad(\rho_A)$. Then we define

$$\Phi(A[X]) = \frac{\{\rho : G \to \text{GL}_2(A[X]) | (\rho \mod X) = \rho_A, [\rho] \in \mathcal{F}(A[X])\}}{1 + M_2(X)},$$

where $[\rho]$ is the isomorphism class in $\mathcal{F}(A[X])$ containing $\rho$ and $\rho$ is assumed to satisfy the lifting property described in §1.22.
§1.24. Cocycles and deformations.
Take $X$ finite as above. For $\rho \in \Phi(A[X])$, we can write $\rho = \rho_A \oplus u'_\rho$ letting $\rho_A$ acts on $M_2(X)$ by matrix multiplication from the right. Then as before

$$\rho_A(gh) \oplus u'_\rho(gh) = (\rho_A(g) \oplus u'_\rho(g))(\rho_A(h) \oplus u'_\rho(h))$$

$$= \rho_A(gh) \oplus (u'_\rho(g)\rho_A(h) + \rho_A(g)u'_\rho(h))$$

produces $u'_\rho(gh) = u'_\rho(g)\rho_A(h) + \rho_A(g)u'_\rho(h)$ and multiplying by $\rho_A(gh)^{-1}$ from the right, we get the cocycle relation for $u_\rho(g) = u'_\rho(g)\rho_A(g)^{-1}$:

$$u_\rho(gh) = u_\rho(g) + gu_\rho(h) \quad \text{for} \quad gu_\rho(h) = \rho(g)u_\rho(h)\rho_A(g)^{-1},$$

getting the map $\Phi(A[X]) \to H^1(G, ad(X))$ which factors through $H^1(G, Ad(X))$. As before this map is injective $A$-linear map identifying $\Phi(A[X])$ with $\text{Sel}(Ad(X))$. 
§1.25. General adjoint Selmer group. We see that \( u_\rho : G \to Ad(X) \) is a 1-cocycle, and we get an embedding \( \Phi(A[X]) \hookrightarrow H^1(G, Ad(X)) \) for \( l \in S \cup \{p\} \) by \( \rho \mapsto [u_\rho] \). The local version of \( \Phi \): 
\[
\Phi_p(A[X]) := \left\{ \rho : D_p \to \text{GL}_2(A[X]) \mid \rho \text{ mod } X = \rho_A, [\rho] \in \mathcal{F}_p(A[X]) \right\} / 1 + M_2(X),
\]
which is identified with \( F^+_H H^1(D_p, Ad(X)) \). Define 
\[
\text{Sel}(Ad(X)) := \text{Ker}(H^1(G, Ad(X)) \to \frac{H^1(D_p, Ad(X))}{F^+_H H^1(D_p, Ad(X))}),
\]
If \( X = \lim_{\to i} X_i \) for finite \( A \)-modules \( X_i \), we just define 
\[
\text{Sel}(Ad(X)) = \lim_{\to i} \text{Sel}(Ad(X_i)).
\]
Then for finite \( X_i \), 
\[
\Phi(A[X_i]) = \text{Sel}(Ad(X_i)) \quad \text{and} \quad \lim_{\to i} \Phi(A[X_i]) = \text{Sel}(\lim_{\to i} Ad(X_i)).
\]
§1.26. **Differentials and Selmer group.** For each \([\rho_A] \in \mathcal{F}(A)\), choose a representative \(\rho_A = \varphi \circ \rho\) as in §1.22. Then we have a map \(\Phi(A[X]) \to \mathcal{F}(A[X])\) for each finite \(A\)-module \(X\) sending \(\rho \in \Phi(A[X])\) chosen as in §1.21 to the class \([\rho] \in \mathcal{F}(A[X])\). By our choice of \(\rho\) as in §1.21, this map is injective.

Conversely pick a class \(c \in \mathcal{F}(A[X])\) over \([\rho_A] \in \mathcal{F}(A)\). Then for \(\rho \in c\), we have \(x \in 1 + M_2(m_A[X])\) such that \(x\rho x^{-1} \mod X = \rho_A\). By replacing \(\rho\) by \(x\rho x^{-1}\) and choosing the lifted base, we conclude \(\Phi(A[X]) \cong \{[\rho] \in \mathcal{F}(A[X])|\rho \mod X \sim \rho_A\}\); so, for finite \(X\),

\[
\text{Sel}(Ad(X)) = \Phi(A[X]) = \{\phi \in \text{Hom}_{B\text{-alg}}(R_X, A[X]) : \phi \mod X = \varphi\} = \text{Der}_B(R_X, X) \cong \text{Hom}_{R_X}(\Omega_{R_X/B}, X) \cong \text{Hom}_A(\Omega_{R_X/B} \otimes_{R_X} \varphi A, X).
\]

Thus

\[
\text{Sel}(Ad(X)) \cong \text{Hom}_A(\Omega_{R_X/B} \otimes_{R_X} \varphi A, X).
\]
1.27. Theorem $\text{Sel}(\text{Ad}(\rho_A))^\vee \cong \Omega_{R_x/B} \otimes_{R_x, \varphi} A$.

Proof. Take the Pontryagin dual

$A^\vee := \text{Hom}_B(A, B^\vee) = \text{Hom}_{\mathbb{Z}_p}(A \otimes B, \mathbb{Q}_p/\mathbb{Z}_p) = \text{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$.

Since $A = \lim_{\leftarrow} A_i$ for finite $i$ and $\mathbb{Q}_p/\mathbb{Z}_p = \lim_{\rightarrow} p^{-1}\mathbb{Z}/\mathbb{Z}$, $A^\vee = \lim_{\rightarrow} \text{Hom}(A_i, \mathbb{Q}_p/\mathbb{Z}_p) = \lim_{\rightarrow} A_i^\vee$ is a union of the finite modules $A_i^\vee$.

We define $\text{Sel}(\text{Ad}(\rho_A)) := \lim_{\rightarrow} \text{Sel}(\text{Ad}(A_i^\vee))$. Defining $\Phi(A[A^\vee]) = \lim_{\rightarrow} \Phi(A[A_i^\vee])$, we see from compatibility of cohomology with injective limit

$$\Phi(A[A^\vee]) = \text{Sel}(\text{Ad}(\rho_A)) = \lim_{\rightarrow} \text{Sel}(\text{Ad}(A_i^\vee))$$

$$= \lim_{\rightarrow} \text{Ker}(H^1(G, \text{Ad}(A_i^\vee)) \to \frac{H^1(D_p, \text{Ad}(A_i^\vee))}{F_+^+ H^1(D_p, \text{Ad}(A_i^\vee))})$$
§1.28. **Proof continues.** By the boxed formula in §1.25,

\[
\text{Sel}(\text{Ad}(\rho_A)) = \lim_{i} \text{Sel}(\text{Ad}(A_i^\vee)) = \lim_{i} \text{Hom}_{R_x}(\Omega_{R_x/B \otimes R_x} A, A_i^\vee) \\
= \text{Hom}_{A}(\Omega_{R_x/B \otimes R_x} A, A^\vee) = \text{Hom}_{A}(\Omega_{R_x/B \otimes R_x} A, \text{Hom}_{\mathbb{Z}_p}(A, \mathbb{Q}_p/\mathbb{Z}_p)) \\
= \text{Hom}_{\mathbb{Z}_p}(\Omega_{R_x/B \otimes R_x} A, \mathbb{Q}_p/\mathbb{Z}_p) = (\Omega_{R_x/B \otimes R_x} A)^\vee.
\]

Taking Pontryagin dual back, we finally get

\[
\text{Sel}(\text{Ad}(\rho_A))^\vee \cong \Omega_{R_x/B \otimes R_x, \varphi} A \quad \text{and} \quad \text{Sel}(\text{Ad}(\rho)) = \Omega_{R_x/B \otimes R_x} \mathbb{F}
\]

as desired. In particular, \(\text{Sel}(\text{Ad}(\rho_\chi))^\vee = \Omega_{R_x/B}\) (with \(\rho_{\kappa} = \rho_{\text{ord}}\) if \(\chi = \kappa\)).

This is the generalization of the formula in §0.19

\[
C_k \cong \Omega_{\mathbb{Z}_p[C_k]/\mathbb{Z}_p \otimes \mathbb{Z}_p[C_k]} \mathbb{Z}_p.
\]
§1.29. $p$-Local condition. The submodule $\Phi_p(A[X])$ in the cohomology group $H^1(\mathbb{Q}_p, \text{Ad}(X))$ is made of classes of 1-cocycles $u$ with $u|_{I_p}$ is upper nilpotent and $u|_{D_p}$ is upper triangular with respect to the compatible basis $(v_p, w_p)$. Suppose we have $\sigma \in I_p$ such that $\rho_A(\sigma) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ such that $\alpha \not\equiv \beta \mod m_A$. Suppose $u$ is upper nilpotent over $I_p$. Then for $\tau \in D_p$, we have $\text{Ad}(\rho_A)(\tau)u(\tau^{-1}\sigma\tau) = (\text{Ad}(\rho_A)(\sigma)-1)u(\tau) + u(\sigma)$. Writing $u(\tau) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, we find $(\text{Ad}(\rho_A)(\sigma) - 1)u(\tau) = \begin{pmatrix} 0 & (\alpha\beta^{-1}-1)b \\ (\alpha^{-1}\beta^{-1}-1)c & 0 \end{pmatrix}$.

Since $\rho_A(\tau)$ is upper triangular and $u(\tau^{-1}\sigma\tau)$ is upper nilpotent, $\text{Ad}(\rho_A)(\tau)u(\tau^{-1}\sigma\tau)$ is still upper nilpotent; so, $(\alpha^{-1}\beta - 1)c = 0$ and hence $c = 0$. Therefore $u$ is forced to be upper triangular over $D_p$. Thus we get

**Lemma 4.** If $\bar{\rho}(\sigma)$ for at least one $\sigma \in I_p$ has two distinct eigenvalues, $\Phi_p(A[X])$ gives rise to the subgroup of $H^1(\mathbb{Q}_p, \text{Ad}(X))$ made of classes containing a 1-cocycle whose restriction to $I_p$ is upper nilpotent.