Lecture slide No.1 for Math 207c
Adjoint Selmer groups

Haruzo Hida

We define Sel($Ad(\rho_A)$) for ordinary deformations $\rho_A \in \mathcal{D}_\chi(A)$ of an absolutely irreducible 2-dimensional minimal Galois representation $\overline{\rho}$ and show that Sel($Ad(\overline{\rho})$) $= t_{R/B}$ and Sel($Ad(\rho_A)$)$^\vee \cong \Omega_{R/B} \otimes_{R,\varphi} A$, where $\varphi : R \to A$ with $\varphi \circ \boldsymbol{\rho} \sim \rho_A$ for the universal minimal ordinary Galois representation $\boldsymbol{\rho} : G \to \mathrm{GL}_2(R)$ of $\overline{\rho}$. Here the deformation functors $\mathcal{D}, \mathcal{D}_\chi : \mathcal{C} \to SETS$ are defined in §0.22.

As before, we write I_l for the inertia group of the l-decomposition subgroup $D_l \subset G$. We write S for the set of ramified primes $l \neq p$ of $\overline{\rho}$ such that $\overline{\rho}|_{I_l} \cong \overline{\epsilon}_l \oplus \overline{\delta}_l$. We set $\mathbb{F}[\varepsilon] := \mathbb{F}[X]/(X^2)$ (dual numbers) with $\varepsilon \leftrightarrow X \mod (X^2)$.

§1.1. p-Ordinarity condition

Fix $\overline{\rho} : G \to \mathrm{GL}_2(\mathbb{F})$ with $\overline{\rho} = \rho_A|_{D_p} \cong \begin{pmatrix} \overline{\epsilon} & * \\ 0 & \overline{\delta} \end{pmatrix}$ and $\overline{\epsilon} \neq \overline{\delta}$. Let $\rho_A : G \to \mathrm{GL}_2(A)$ $(A \in \mathcal{C})$ be a deformation of $\overline{\rho} : G \to \mathrm{GL}_2(\mathbb{F})$ acting on $V(\rho_A)$. We say ρ is p-ordinary if
(ord$_p$) $\rho_A|_{D_p} \cong \begin{pmatrix} \epsilon_A & * \\ 0 & \delta_A \end{pmatrix}$ for two characters $\epsilon_A, \delta_A : D_p \to A^\times$ distinct modulo $\mathfrak{m}_A$ with δ_A unramified with $\delta_A \bmod \mathfrak{m}_A = \overline{\delta}$ (this is a requirement called p-distinguishedness).

Since twisting by a character $\xi : G \to B^\times$ induces isomorphism between the functors deforming $\overline{\rho}$ and $\overline{\rho} \otimes \xi$, we may assume a similar condition for $l \in S$ $(l \neq p)$:
(ord$_l$) $\rho|_{I_l} \cong \begin{pmatrix} \epsilon_{l,A} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ with $\epsilon_{l,A} \neq \mathbf{1}$.

We can fix a character $\chi : G \to B^\times$, we consider
(det) $\det \rho = \iota_A \circ \chi$ for the B-algebra structure $\iota_A : B \to A$.
The fixed determinant functor is denoted by $\mathcal{D}_\chi : \mathcal{C} \to SETS$.

§1.2. Deformation functor.

We consider the following functors for a fixed absolutely irreducible representation $\overline{\rho} : G \to \mathrm{GL}_2(\mathbb{F})$ satisfying (ord_p) and (ord_l). Recall $\mathcal{D}^{\emptyset}, \mathcal{D}, \mathcal{D}_\chi : \mathcal{C} \to SETS$ given by

$$\mathcal{D}^{\emptyset}(A) := \{\rho_A : G \to \mathrm{GL}_2(A) | \rho_A \bmod \mathfrak{m}_A = \overline{\rho}\}/\Gamma(\mathfrak{m}_A),$$
$$\mathcal{D}(A) = \{\rho_A \in \mathcal{D}^{\emptyset}(A) | (\mathrm{min}),\ (\mathrm{ord}_p) \text{ and } (\mathrm{ord}_l)\},$$
$$\mathcal{D}_\chi(A) = \{\rho_A \in \mathcal{D}(A) | \det \rho = \iota_A \circ \chi\}.$$

Then

Theorem 1 (B. Mazur)**.** *There exists universal couples $(R, \boldsymbol{\rho})$, $(R^{ord}, \boldsymbol{\rho}^{ord})$ and $(R_\chi, \boldsymbol{\rho}_\chi)$ representing $\mathcal{D}^{\emptyset}$, $\mathcal{D}$ and $\mathcal{D}_\chi$, respectively, so that $\mathcal{D}(A) \cong \mathrm{Hom}_{\mathcal{C}}(R^{ord}, A)$ by $\rho \mapsto \varphi$ with $\varphi \circ \boldsymbol{\rho}^{ord} \sim \rho$ (resp. $\mathcal{D}_\chi(A) \cong \mathrm{Hom}_{\mathcal{C}}(R_\chi, A)$ by $\rho \mapsto \varphi$ with $\varphi \circ \boldsymbol{\rho}_\chi \sim \rho$).*

We admit this theorem (see [MFG, §2.3] or Mazur's paper quoted there).

§1.3. Fiber products.

Let $C = \mathcal{C}, SETS$. For arrows $\phi' : S' \to S$ and $\phi'' : S'' \to S$ in C,

$$S' \times_S S'' = \{(a', a'') \in S' \times S'' | \phi'(a') = \phi''(a'')\}$$

gives the fiber product of S' and S'' over S in C. So

$$\mathrm{Hom}_C(X, S' \times_S S'') = \mathrm{Hom}_C(X, S') \times_{\mathrm{Hom}_C(X,S)} \mathrm{Hom}_C(X, S'')$$

for any $X \in C$. Let $\mathcal{F} : C \to SETS$ be a covariant functor. We assume

$$|\mathcal{F}(\mathbb{F})| = 1 \text{ and } \mathcal{F}(\mathbb{F}[\varepsilon] \times_{\mathbb{F}} \mathbb{F}[\varepsilon]) = \mathcal{F}(\mathbb{F}[\varepsilon]) \times_{\mathcal{F}(\mathbb{F})} \mathcal{F}(\mathbb{F}[\varepsilon])$$

by two projections.

It is easy to see $\mathcal{F} \in \{\mathcal{D}^{\emptyset}, \mathcal{D}, \mathcal{D}_{\chi}\}$ satisfies this condition. Indeed, noting that $\mathbb{F}[\varepsilon] \times_{\mathbb{F}} \mathbb{F}[\varepsilon] \cong \mathbb{F}[\varepsilon'] \times_{\mathbb{F}} \mathbb{F}[\varepsilon''] \cong \mathbb{F}[\varepsilon', \varepsilon'']$, if $\rho' \in \mathcal{F}(\mathbb{F}[\varepsilon'])$ and $\rho'' \in \mathcal{F}(\mathbb{F}[\varepsilon''])$, we have $\rho' \times \rho''$ has values in $\mathrm{GL}_2(\mathbb{F}[\varepsilon', \varepsilon''])$ is an element in $\mathcal{F}(\mathbb{F}[\varepsilon'] \times_{\mathbb{F}} \mathbb{F}[\varepsilon''])$.

§1.4. Tangent space of deformation functors.

For $A \in \mathcal{C}$ and an A-module X, suppose

$$|\mathcal{F}(A)| = 1 \text{ and } \mathcal{F}(A[X] \times_A A[X]) = \mathcal{F}(A[X]) \times_{\mathcal{F}(A)} \mathcal{F}(A[X]).$$

Note $A[X] \times_A A[X] = A[X \oplus X]$. The addition on X and A-linear map $\alpha : X \to X$ induces in the same way $\mathcal{C}$-morphisms $+_* : A[X \oplus X] \to A[X]$ by $a + (x \oplus y) \mapsto a + x + y$ and $\alpha_* : A[X] \to A[X]$ by $a + x \mapsto a + \alpha(x)$. Thus we have by functoriality. the "addition"

$$+ : \mathcal{F}(A[X]) \times_{\mathcal{F}(A)} \mathcal{F}(A[X]) = \mathcal{F}(A[X \oplus X]) \xrightarrow{\mathcal{F}(+_*)} \mathcal{F}(A[X])$$

and α-action

$$\alpha : \mathcal{F}(A[X]) \xrightarrow{\mathcal{F}(\alpha_*)} \mathcal{F}(A[X]).$$

With $\mathbf{0} = \mathrm{Im}(\mathcal{F}(A) \to \mathcal{F}(A[X])$ for the inclusion $A \hookrightarrow A[X]$, this makes $\mathcal{F}(A[X])$ as an A-module; so, $\mathcal{F}(\mathbb{F}[\varepsilon])$ is an $\mathbb{F}$-vector space (called the tangent space of $\mathcal{F}$).

§1.5. Cotangent spaces of local rings

Suppose that B is noetherian and pick $R \in CL_B$.

Lemma 1. *The ring R is noetherian if and only if $t^*_{R/W} = \mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_B)$ is a finite dimensional vector space over $\mathbb{F}$.*

The space $t^*_{R/B}$ is called the co-tangent space of R at $\mathfrak{m}_R = (\varpi) \in \text{Spec}(R)$ over $\text{Spec}(B)$. If $\mathfrak{m}_B = (x_1, \ldots, x_r)$, then $\mathfrak{m}_B^n/\mathfrak{m}_B^{n+1}$ is generated by degree n monomial of x_j; so, $B/\mathfrak{m}_B^n$ is generated by degree $\leq n$ polynomial of x_j. Thus for $W = W(\mathbb{F})$, $W[X_1, \ldots, X_r]$ has dense image in B by sending x_j to X_j, and hence $W[[X_1, \ldots, X_r]] \twoheadrightarrow B$.

Since we have an exact sequence: $\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_R/\mathfrak{m}_R^2 \twoheadrightarrow t^*_{R/W}$, we conclude in the same way that $W[[X_1, \ldots, X_r, X_{r+1}, \ldots, X_{r+s}]]$ surjects onto R sending X_i with $i > r$ to generators of $t^*_{R/B}$. Thus the number of generators over B of R is $\dim_{\mathbb{F}} t^*_{R/B}$.

§1.6. Adjoint Galois modules

Let $M_2(A)$ be the space of 2×2 matrices with coefficients in A. We let G acts on $M_2(A)$ by $gv = \rho_A(g) v \rho_A(g)^{-1}$. This action is called the **adjoint** action of G, and this G–module will be written as $ad(\rho_A)$.

Write Z for the center of $M_2(A)$ (scalar matrices) and define $\mathfrak{sl}_2(A) = \{X \in M_2(A) | \mathrm{Tr}(X) = 0\}$. Since $\mathrm{Tr}(aXa^{-1}) = \mathrm{Tr}(X)$, $\mathfrak{sl}_2(A)$ is stable under the adjoint action. This Galois module will be written as $Ad(\rho_A)$.

Since $p > 2$, $X \mapsto \frac{1}{2}\mathrm{Tr}(X) \oplus (X - \frac{1}{2}\mathrm{Tr}(X))$ gives rise to $M_2(A) = Z \oplus \mathfrak{sl}_2(A)$ stable under the adjoint action.

So we have $ad(\rho_A) = \mathbf{1} \oplus Ad(\rho_A)$, where $\mathbf{1}$ is the trivial representation.

§1.7. Tangent space as cohomology

Lemma 2. *Let $(R, \boldsymbol{\rho})$ be the universal couple representing $\mathcal{D}^{\emptyset}$ over CL_W. Then*

$$t_{R/W} := \mathrm{Hom}_{\mathbb{F}}(t^*_{R/W}, \mathbb{F}) \cong H^1(G, ad(\overline{\rho})),$$

where $H^1(G, ad(\overline{\rho}))$ is the continuous first cohomology group of G with coefficients in the discrete G–module $V(ad(\overline{\rho}))$.

Proof, Step. 1, dual number.
We claim: $\mathrm{Hom}_{CL_W}(R, \mathbb{F}[\varepsilon]) \cong t_{R/W}$. Construction of the map.

Start with a W-algebra homomorphism $\phi : R \to \mathbb{F}[\varepsilon]$. Write

$$\phi(r) = \phi_0(r) + \phi_\varepsilon(r)\varepsilon \quad \text{with } \phi_0(r), \phi_\varepsilon(r) \in \mathbb{F}.$$

Then the map is $\phi \mapsto \ell_\phi = \phi_\varepsilon|_{\mathfrak{m}_R}$.

§1.8. Step. 2, Well defined-ness of ℓ_ϕ

From $\phi(ab) = \phi(a)\phi(b)$, we get

$$\phi_0(ab) = \phi_0(a)\phi_0(b) \text{ and } \phi_\varepsilon(ab) = \phi_0(a)\phi_\varepsilon(b) + \phi_0(b)\phi_\varepsilon(a).$$

Thus $\phi_\varepsilon \in Der_W(R, \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}(\Omega_{R/W} \otimes_R \mathbb{F}, \mathbb{F})$. Since for any derivation $\delta \in Der_W(R, \mathbb{F})$, $\phi' = \phi_0 + \delta\varepsilon \in \mathrm{Hom}_{CL_W}(R, \mathbb{F}[\varepsilon])$,

$$\begin{aligned}\mathrm{Hom}_R(\Omega_{R/W} \otimes_R \mathbb{F}, \mathbb{F}) &\cong \mathrm{Hom}_R(\Omega_{R/W}, \mathbb{F}) \\ &\cong Der_W(R, \mathbb{F}[\varepsilon]) \cong \mathrm{Hom}_{CL_W}(R, \mathbb{F}[\varepsilon]).\end{aligned}$$

Note $\mathrm{Ker}(\phi_0) = \mathfrak{m}_R$ because R is local. Since ϕ is W-linear, $\phi_0(a) = \overline{a} = a \mod \mathfrak{m}_R$. Thus ϕ kills $\mathfrak{m}_R^2$ and takes $\mathfrak{m}_R$ W-linearly into $\mathfrak{m}_{\mathbb{F}[\varepsilon]} = \mathbb{F}\varepsilon$; so, $\ell_\phi : t_R^* := \mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathbb{F}$. For $r \in W$, $\overline{r} = r\phi(1) = \phi(r) = \overline{r} + \phi_\varepsilon(r)\varepsilon$, and hence ϕ_ε kills W; so, $\ell_\phi \in t_{R/W}$.

§1.9. Step. 3, $\phi \mapsto \ell_\phi$ is an injection.

Since R shares its residue field $\mathbb{F}$ with W, any element $a \in R$ can be written as $a = r + x$ with $r \in W$ and $x \in \mathfrak{m}_R$.

Thus ϕ is completely determined by the restriction ℓ_ϕ of ϕ_ε to $\mathfrak{m}_R$, which factors through $t^*_{R/W}$.

Thus $\phi \mapsto \ell_\phi$ induces an injective linear map $\ell : \mathrm{Hom}_{W-alg}(R, \mathbb{F}[\varepsilon]) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}(t^*_{R/W}, \mathbb{F})$.

Note $R/(\mathfrak{m}_R^2 + \mathfrak{m}_W) = \mathbb{F} \oplus t^*_{R/W} = \mathbb{F}[t^*_{R/W}]$ with the projection $\pi : R \twoheadrightarrow t^*_{R/W}$ to the direct summand $t^*_{R/W}$. Indeed, writing $\overline{r} = (r \mod \mathfrak{m}_R)$, for the inclusion $\iota : \mathbb{F} = W/\mathfrak{m}_W \hookrightarrow R/(\mathfrak{m}_R^2 + m_W)$, $\pi(r) = r - \iota(\overline{r})$.

§1.10. Step. 4, $\phi \mapsto \ell_\phi$ is a surjection.
For any $\ell \in \mathrm{Hom}_{\mathbb{F}}(t^*_{R/W}, \mathbb{F})$, we extends ℓ to R by putting $\ell(r) = \ell(\pi(r))$. Then we define $\phi : R \to \mathbb{F}[\varepsilon]$ by $\phi(r) = \overline{r} + \ell(\pi(r))\varepsilon$. Since $\varepsilon^2 = 0$ and $\pi(r)\pi(s) = 0$ in $\mathbb{F}[t^*_{R/W}]$, we have

$$rs = (\overline{r} + \pi(r))(\overline{s} + \pi(s)) = \overline{rs} + \overline{s}\pi(r) + \overline{r}\pi(s)$$
$$\xrightarrow{\phi} \overline{rs} + \overline{s}\ell(\pi(r))\varepsilon + \overline{r}\ell(\pi(s))\varepsilon = \phi(r)\phi(s)$$

is an $W-$algebra homomorphism. In particular, $\ell(\phi) = \ell$, and hence ℓ is surjective.

By $\mathrm{Hom}_R(\Omega_{R/W} \otimes_R \mathbb{F}, \mathbb{F}) \cong \mathrm{Hom}_{CL_W}(R, \mathbb{F}[\varepsilon])$, we have

$$\mathrm{Hom}_R(\Omega_{R/W} \otimes_R \mathbb{F}, \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}(t^*_{R/W}, \mathbb{F});$$

so, if $t^*_{R/W}$ is finite dimensional, we also get

$$\boxed{\Omega_{R/W} \otimes_R \mathbb{F} \cong t^*_{R/W}.}$$

§1.11. Step. 5, use of universality.

By the universality, we have

$$\mathrm{Hom}_{CL_B}(R, \mathbb{F}[\varepsilon]) \cong \{\rho : G \to GL_2(\mathbb{F}[\varepsilon]) | \rho \mod \mathfrak{m}_{\mathbb{F}[\varepsilon]} = \overline{\rho}\}/\sim .$$

Write $\rho(g) = \overline{\rho}(g) + u'_\phi(g)\varepsilon$ for ρ corresponding to $\phi : R \to \mathbb{F}[\varepsilon]$. From the mutiplicativity, we have

$$\begin{aligned}\overline{\rho}(gh) + u'_\phi(gh)\varepsilon = \rho(gh) = \rho(g)\rho(h) \\ = \overline{\rho}(g)\overline{\rho}(h) + (\overline{\rho}(g)u'_\phi(h) + u'_\phi(g)\overline{\rho}(h))\varepsilon,\end{aligned}$$

Thus as a function $u' : G \to M_n(\mathbb{F})$, we have

$$u'_\phi(gh) = \overline{\rho}(g)u'_\phi(h) + u'_\phi(g)\overline{\rho}(h). \tag{1}$$

§1.12. Step. 6, Getting 1-cocycle.

Define a map $u_\rho = u_\phi : G \to ad(\overline{\rho})$ by

$$u_\phi(g) = u'_\phi(g)\overline{\rho}(g)^{-1}.$$

Then by a simple computation, we have

$$gu_\phi(h) = \overline{\rho}(g)u_\phi(h)\overline{\rho}(g)^{-1}$$

from the definition of $ad(\overline{\rho})$. Then from the above formula (1), we conclude that

$$\boxed{u_\phi(gh) = gu_\phi(h) + u_\phi(g).}$$

Thus $u_\phi : G \to ad(\overline{\rho})$ is a 1–cocycle. Thus we get an $\mathbb{F}$-linear map

$$t_{R/W} \cong \mathrm{Hom}_{CL_W}(R, \mathbb{F}[\varepsilon]) \to H^1(G, ad(\overline{\rho}))$$

by $\ell_\phi \mapsto [u_\phi]$

§1.13. Step. 7, End of proof.

By computation, for $x \in ad(\overline{\rho})$

$$\rho \sim \rho' \Leftrightarrow \overline{\rho}(g) + u'_\rho(g)\varepsilon = (1 + x\varepsilon)(\overline{\rho}(g) + u'_{\rho'}(g)\varepsilon)(1 - x\varepsilon)$$
$$\Leftrightarrow u'_\rho(g) = x\overline{\rho}(g) - \overline{\rho}(g)x + u'_{\rho'}(g)$$
$$\Leftrightarrow u_\rho(g) = (1 - g)x + u_{\rho'}(g).$$

Thus the cohomology classes of u_ρ and $u_{\rho'}$ are equal if and only if $\rho \sim \rho'$. This shows:

$$\mathrm{Hom}_{\mathbb{F}}(t^*_{R/W}, \mathbb{F}) \cong \mathrm{Hom}_{W-alg}(R, \mathbb{F}[\varepsilon]) \cong$$
$$\{\rho : G \to GL_2(\mathbb{F}[\varepsilon]) | \rho \mod \mathfrak{m}_{\mathbb{F}[\varepsilon]} = \overline{\rho}\}/\sim$$
$$\cong H^1(G, ad(\overline{\rho})).$$

In this way, we get a bijection between $\mathrm{Hom}_{\mathbb{F}}(t^*_{R/W}, \mathbb{F})$ and $H^1(G, ad(\overline{\rho}))$.

§1.14. Tangent space of rings and deformation functor

Lemma 3. *Let $\mathcal{F} = \mathcal{D}^{\emptyset}, \mathcal{D}, \mathcal{D}_{\chi}$ and R, R^{ord} or R_{χ} accordingly. Then $t_{R/B} \cong \mathcal{F}(\mathbb{F}[\varepsilon])$ as $\mathbb{F}$-vetor spaces.*

Proof. Let R be the universal ring for $\mathcal{D}^{\emptyset}$. We have got a canonical bijection in §1.7:

$$\mathcal{D}^{\emptyset}(\mathbb{F}[\varepsilon]) \xrightarrow[i_1]{\text{1-1 onto}} H^1(G, ad(\overline{\rho})) \xrightarrow[i]{\sim} t_{R/B}$$

with a vector space isomorphism i. We have constructed a cocycle u_ρ from $\rho \in \mathcal{F}(\mathbb{F}[\varepsilon])$ writing $\rho = \overline{\rho} + u_\rho \overline{\rho}\varepsilon$. Regarding $(\rho, \rho') \in \mathcal{F}(\mathbb{F}[\varepsilon]) \times \mathcal{F}(\mathbb{F}[\varepsilon]) = \mathcal{F}(\mathbb{F}[\varepsilon] \times_{\mathbb{F}} \mathbb{F}[\varepsilon])$, we see that $+(\rho, \rho') = \overline{\rho} + (u_\rho \overline{\rho} + u_{\rho'} \overline{\rho})\varepsilon \in \mathcal{F}(\mathbb{F}[\varepsilon])$; so, i_1 is a homomorphism. Similarly, one can check that it is $\mathbb{F}$-linear. Same for R^{ord} and R_{χ}. □

§1.15. Galois deformation ring is noetherian.

Let $H = \mathrm{Gal}(F^{(p)}(\overline{\rho})/F(\overline{\rho}))$. Note that $H^{ab} = C_{F(\overline{\rho})}(p^\infty) = \varprojlim_n Cl_{F(\overline{\rho})}(p^\infty)/Cl_{F(\overline{\rho})}(p^\infty)^{p^n}$ and we have an exact sequence for the integer ring O of $F(\overline{\rho})$:

$$\widehat{O}_p^\times \to H^{ab} \to C_{F(\overline{\rho})} \to 1.$$

Therefore H^{ab} is a $\mathbb{Z}_p$-module of finite type, which tells us finiteness of $\mathrm{Hom}(H^{ab}, ad(\overline{\rho}))$. By inflation-restriction sequence,

$$0 \to H^1(F(\overline{\rho})/\mathbb{Q}, ad(\overline{\rho})) \to H^1(G, ad(\overline{\rho})) \to \mathrm{Hom}(H^{ab}, ad(\overline{\rho}))$$

is exact. Since $[F(\overline{\rho}) : \mathbb{Q}] < \infty$ and $|ad(\overline{\rho})| < \infty$, $H^1(F(\overline{\rho})/\mathbb{Q}, ad(\overline{\rho}))$ is finite. Thus $H^1(G, ad(\overline{\rho})) \cong t_{R/W}$ is finite. Then by the lemma in §1.14, R is noetherian. This also tells us that R^{ord} and R_χ are noetherian.

§1.16 Tangent space with local condition.

We regard $\mathcal{F}(\mathbb{F}[\varepsilon]) \subset H^1(G, ad(\overline{\rho}))$. We may choose by (ord$_p$) a basis (dependent on $l \in S \cup \{p\}$) of $V(\rho)$ for $\rho \in \mathcal{F}(\mathbb{F}[\varepsilon])$ so that $\rho|_{D_p}$ is upper triangular with quotient character δ congruent to $\overline{\delta}$ modulo $\mathfrak{m}_A$. Similarly by (ord$_l$), we choose the basis so that $\rho|_{I_l} = \epsilon_l \oplus \mathbf{1}$ in this order.

Theorem 2. *A 1-cocycle u gives rise to a class in $\mathcal{D}_\chi(\mathbb{F}[\varepsilon])$ if and only if $u|_{D_p}$ is upper triangular, $u|_{I_p}$ is upper nilpotent and* $\mathrm{Tr}(u) = 0$ *over G, where $\overline{v} = v \mod (\epsilon)$.*

For primes $l \neq p$, $u(I_l) = 0$ as $p \nmid |I_l|$ (minimality). The description of cocycles u is independent of χ; so, the tangent space $t_{R_\chi/B}$ is independent as a cohomology subgroup as long as $\mathbb{F}$ does not change.

§**1.17. Proof**.

By (det), $1 = \det(\rho\overline{\rho}^{-1}) = 1 + u_\rho\varepsilon = 1 + \mathrm{Tr}(u_\rho)\varepsilon$; so, (det) $\Leftrightarrow$ $\mathrm{Tr}(u) = 0$ over G. Thus we $t_{R_\chi/B} \subset H^1(G, Ad(\overline{\rho}))$.

Choose a generator $w \in V(\epsilon)$ over $\mathbb{F}[\varepsilon]$. Then (w, v) is a basis of $V(\rho)$ over $\mathbb{F}[\varepsilon]$. Let $(\overline{w}, \overline{v}) = (w, v) \bmod \varepsilon$ and identify $V(ad(\overline{\rho}))$ with $M_2(\mathbb{F})$ with this basis. Then defining $\overline{\rho}$ by $(\sigma\overline{w}, \sigma\overline{v}) = (\overline{w}, \overline{v})\overline{\rho}(\sigma)$, for $\sigma \in D_p$, we have $\overline{\rho}(\sigma) = \begin{pmatrix} \overline{\epsilon}(\sigma) & * \\ 0 & \overline{\delta}(\sigma) \end{pmatrix}$ (upper triangular). If $\sigma \in I_p$, $\rho\overline{\rho}^{-1} = 1 + u_\rho$ with lower right corner of u_ρ has to vanish as $\delta = 1$ on I_p, we have $u_\rho(\sigma) \in \{\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)\}$.

The condition (ord_p) is equivalent to u_ρ is of the form $\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$ but by $\mathrm{Tr}(u_\rho) = 0$, it has to be upper nilpotent; i.e., $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$. $\square$

§**1.18. Adjoint Selmer group.** For $\mathcal{F} = \mathcal{D}$ or $\mathcal{D}_\chi$, we define the local deformation functor $\mathcal{D}_{\chi,p}$ by sending A to

$$\{\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(A) | \rho_A \bmod \mathfrak{m}_A = \overline{\rho} \text{ and } (\mathrm{ord}_p) \text{ and } (\det)\}.$$

By the proof of the theorem in §1.16, $\mathcal{D}_{\chi,p}(\mathbb{F}[\varepsilon])$ is the space of cohomology classes in $H^1(D_p, Ad(\overline{\rho}))$ upper triangular over D_p and upper nilpotent over I_p. Define $Ad(\rho_A)$ by the conjugation action on $\mathfrak{sl}_2(A)$ by ρ_A, and put $Ad(\rho_A)^* := Ad(\rho_A) \otimes_A A^\vee$ (discrete), writing $A^\vee = \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ (Pontryagin dual). Define

$$\mathrm{Sel}(Ad(\rho_A)) := \mathrm{Ker}(H^1(G, Ad(\rho_A)^*) \to \frac{H^1(D_p, Ad(\rho_A)^*)}{F_-^+ H^1(D_p, Ad(\rho_A)^*)}),$$

where $F_-^+ H^1(D_p, Ad(\rho_A)^*) \subset H^1(D_p, Ad(\rho_A)^*)$ is made of cohomology classes upper triangular over D_p and upper nilpotent over I_p. Then we have $\boxed{\mathrm{Sel}(Ad(\overline{\rho})) := t_{R_\chi/B}.}$

§1.19. R^{ord} is an algebra over the Iwasawa algebra

The finite order character $\det(\overline{\rho})$ factors through $\mathrm{Gal}(\mathbb{Q}[\mu_{N_0}]/\mathbb{Q})$ for some positive integer N_0. Let N_0 be the minimal such integer (called conductor of $\det(\overline{\rho})$). Write $N_0 = Np^{\nu}$ for N prime to p; so, N is the prime to p-conductor of $\det(\overline{\rho})$.

If ρ_A is a minimal deformation of $\overline{\rho}$, then $\rho_A(I_l) \cong \overline{\rho}(I_l)$ and hence $\det(\rho_A)(I_l) = \det(\overline{\rho})(I_l)$. Therefore, $\det(\boldsymbol{\rho}^{ord})$ is a minimal deformation of $\det(\overline{\rho})$.

By universality, for the universal character $\boldsymbol{\kappa} : G \to W[[\Gamma]]^{\times}$, we have a (unique) algebra homomorphism $i = i_{R^{ord}} : W[[\Gamma]] \to R^{ord}$ such that $i_{R^{ord}} \circ \boldsymbol{\kappa} = \det(\boldsymbol{\rho})$. Therefore

R^{ord} is canonically an algebra over $\Lambda = W[[\Gamma]]$.

§1.20. Reinterpretation of $\mathcal{D}$

Consider the following deformation functor $\mathcal{D}_{\boldsymbol{\kappa}} : CL_{/\Lambda} \to SETS$

$$\mathcal{D}_{\boldsymbol{\kappa}}(A) = \{\rho : G \to \mathrm{GL}_2(A) | \rho \mod \mathfrak{m}_A \cong \overline{\rho},$$
$$\rho \text{ satisfies } (\mathrm{ord}_p), (\mathrm{ord}_l) \text{ and } (\det{}_\Lambda)\}/\cong,$$

where writing $i_A : \Lambda \to A$ for Λ-algebra structure of A,
$(\det_\Lambda)$ $\det(\rho) = i_A \circ \boldsymbol{\kappa}$.

Proposition 1. *We have* $\mathcal{D}_{\boldsymbol{\kappa}}(A) \cong \mathrm{Hom}_{CL_\Lambda}(R^{ord}, A)$ *with universal representation* $\boldsymbol{\rho}^{ord} \in \mathcal{D}(R^{ord})$*; so,*

$$\mathrm{Sel}(Ad(\overline{\rho})) := t_{R^{ord}/\Lambda} = \mathrm{Ker}(H^1(G, Ad(\overline{\rho})) \to \frac{H^1(D_p, Ad(\overline{\rho}))}{F_-^+ H^1(D_p, Ad(\overline{\rho}))}.$$

§**1.21. Proof.** For any $\rho_A \in \mathcal{D}_{\boldsymbol{\kappa}}(A)$, regard $\rho_A \in \mathcal{D}(A)$. Then we have $\varphi \in \mathrm{Hom}_{\mathcal{C}}(R^{ord}, A)$ such that $\varphi \circ \boldsymbol{\rho}^{ord} \cong \rho_A$. Thus $\varphi \circ \det(\boldsymbol{\rho}^{ord}) = \det(\rho_A)$. Since $\det(\rho_A) = \iota_A \circ \boldsymbol{\kappa}$ and $\det(\boldsymbol{\rho}^{ord}) = \iota_{R^{ord}} \circ \boldsymbol{\kappa}$, we find $\varphi \circ \iota_{R^{ord}} = \iota_A$, and hence $\varphi \in \mathrm{Hom}_{CL_\Lambda}(R^{ord}, A)$. This shows that R^{ord} also represents $\mathcal{D}_{\boldsymbol{\kappa}}$ over Λ.

As we already remarked, $\mathcal{D}_{\boldsymbol{\kappa}}(\mathbb{F}[\varepsilon]) = t_{R^{ord}/\Lambda} = \mathfrak{m}_{R^{ord}}/\mathfrak{m}^2_{R^{ord}} + \mathfrak{m}_\Lambda$ is independent as a subgroup of $H^1(G, Ad(\overline{\rho}))$; so, we get a new expression of $\mathrm{Sel}(Ad(\overline{\rho}))$. □

By the proof, $\Omega_{R^{ord}/\Lambda} \otimes_{R^{ord}} \mathbb{F} \cong \mathrm{Sel}(Ad(\overline{\rho})) \cong \Omega_{R_\chi/B} \otimes_{R_\chi} \mathbb{F}$, so the smallest number of generators of $\Omega_{R^{ord}/\Lambda}$ as R^{ord}-modules and $\Omega_{R_\chi/B}$ as R_χ modules is equal. In the same way, the number of generators of R^{ord} as Λ-algebras and R_χ as B-algebras is equal.

§**1.22. Recall the compatible choice of ρ_A.** By (ord$_l$) for $l \in S \cup \{p\}$, the universal representation $\boldsymbol{\rho}_\chi$ is equipped with a basis $(\mathbf{v}_l, \mathbf{w}_l)$ so that the matrix representation with respect this basis satisfies (ord$_l$). By universality, each class $c \in \mathcal{D}_\chi(A)$ has ρ such that $V(\rho) = V(\boldsymbol{\rho}_\chi) \otimes_{R_\chi,\varphi} A$ for a unique $\varphi \in \mathrm{Hom}_{CL_B}(R_\chi, A)$, we can choose a unique $\rho_A \in c$ with a basis $\{(v_l = \mathbf{v}_l \otimes 1, w_l = \mathbf{w}_l \otimes 1)\}_l$ satisfying $\{(\mathrm{ord}_l){:}l \in S \cup \{p\}\}$ compatible with specialization. We choose such a specific representative ρ_A for each $c \in \mathcal{D}_\chi(A)$.

Start with ρ_A as above. Take a finite A-module X and consider the ring $A[X] = A \oplus X$ with $X^2 = 0$. Then $A[X]$ is still p-profinite. Pick $\rho \in \mathcal{F}(A[X])$ such that $\rho \mod X \sim \rho_A$. By our choice of representative ρ and ρ_A as above, we may (and do) assume $\rho \mod X = \rho_A$.

§**1.23. General cocycle construction.** Here we allow $\chi = \boldsymbol{\kappa}$ but if $\chi = \boldsymbol{\kappa}$. Letting $B = W$ if χ has values in $W^\times$ and Λ if $\chi = \boldsymbol{\kappa}$, the functor $\mathcal{F} = \mathcal{D}_\chi$ is defined over CL_B. Let ρ_A act on $M_2(A)$ and $\mathfrak{sl}_2(A) = \{x \in M_2(A) | \mathrm{Tr}(x) = 0\}$ by conjugation. Write this representation $ad(\rho_A)$ and $Ad(\rho_A)$ as before. Let $ad(X) = ad(\rho_A) \otimes_A X$ and $Ad(X) = Ad(\rho_A) \otimes_A X$ and regard them as G-modules by the action on $ad(\rho_A)$ and $Ad(\rho_A)$. Then we define

$$\Phi(A[X]) = \frac{\{\rho : G \to \mathrm{GL}_2(A[X]) | (\rho \mod X) = \rho_A, [\rho] \in \mathcal{F}(A[X])\}}{1 + M_2(X)},$$

where $[\rho]$ is the isomorphism class in $\mathcal{F}(A[X])$ containing ρ and ρ is assumed to satisfy the lifting property described in §1.22.

§1.24. Cocycles and deformations.

Take X finite as above. For $\rho \in \Phi(A[X])$, we can write $\rho = \rho_A \oplus u'_\rho$ letting ρ_A acts on $M_2(X)$ by matrix multiplication from the right. Then as before

$$\begin{aligned}\rho_A(gh) \oplus u'_\rho(gh) &= (\rho_A(g) \oplus u'_\rho(g))(\rho_A(h) \oplus u'_\rho(h)) \\ &= \rho_A(gh) \oplus (u'_\rho(g)\rho_A(h) + \rho_A(g)u'_\rho(h))\end{aligned}$$

produces $u'_\rho(gh) = u'_\rho(g)\rho_A(h) + \rho_A(g)u'_\rho(h)$ and multiplying by $\rho_A(gh)^{-1}$ from the right, we get the cocycle relation for $u_\rho(g) = u'_\rho(g)\rho_A(g)^{-1}$:

$$u_\rho(gh) = u_\rho(g) + gu_\rho(h) \quad \text{for } gu_\rho(h) = \rho(g)u_\rho(h)\rho_A(g)^{-1},$$

getting the map $\Phi(A[X]) \to H^1(G, ad(X))$ which factors through $H^1(G, Ad(X))$. As before this map is injective A-linear map identifying $\Phi(A[X])$ with $\mathrm{Sel}(Ad(X))$.

§**1.25. General adjoint Selmer group.** We see that $u_\rho : G \to Ad(X)$ is a 1-cocycle, and we get an embedding $\Phi(A[X]) \hookrightarrow H^1(G, Ad(X))$ for $l \in S \cup \{p\}$ by $\rho \mapsto [u_\rho]$. The local version of Φ:

$$\Phi_p(A[X]) := \frac{\{\rho : D_p \to \mathrm{GL}_2(A[X]) | \rho \bmod X = \rho_A, [\rho] \in \mathcal{F}_p(A[X])\}}{1 + M_2(X)},$$

which is identified with $F_-^+ H^1(D_p, Ad(X))$. Define

$$\mathrm{Sel}(Ad(X)) := \mathrm{Ker}(H^1(G, Ad(X)) \to \frac{H^1(D_p, Ad(X))}{F_-^+ H^1(D_p, Ad(X))}),$$

If $X = \varinjlim_i X_i$ for finite A-modules X_i, we just define

$$\mathrm{Sel}(Ad(X)) = \varinjlim_i \mathrm{Sel}(Ad(X_i)).$$

Then for finite X_i,

$$\Phi(A[X_i]) = \mathrm{Sel}(Ad(X_i)) \quad \text{and} \quad \varinjlim_i \Phi(A[X_i]) = \mathrm{Sel}(\varinjlim_i Ad(X_i)).$$

§**1.26. Differentials and Selmer group.** For each $[\rho_A] \in \mathcal{F}(A)$, choose a representative $\rho_A = \varphi \circ \boldsymbol{\rho}$ as in §1.22. Then we have a map $\Phi(A[X]) \to \mathcal{F}(A[X])$ for each finite A-module X sending $\rho \in \Phi(A[X])$ chosen as in §1.21 to the class $[\rho] \in \mathcal{F}(A[X])$. By our choice of ρ as in §1.21, this map is injective.

Conversely pick a class $c \in \mathcal{F}(A[X])$ over $[\rho_A] \in \mathcal{F}(A)$. Then for $\rho \in c$, we have $x \in 1 + M_2(\mathfrak{m}_{A[X]})$ such that $x\rho x^{-1} \mod X = \rho_A$. By replacing ρ by $x\rho x^{-1}$ and choosing the lifted base, we conclude $\Phi(A[X]) \cong \{[\rho] \in \mathcal{F}(A[X]) | \rho \mod X \sim \rho_A\}$; so, for finite X,

$$\mathrm{Sel}(Ad(X)) = \Phi(A[X]) = \{\phi \in \mathrm{Hom}_{B\text{-alg}}(R_\chi, A[X]) : \phi \mathrm{mod}\ X = \varphi\}$$
$$= Der_B(R_\chi, X) \cong \mathrm{Hom}_{R_\chi}(\Omega_{R_\chi/B}, X) \cong \mathrm{Hom}_A(\Omega_{R_\chi/B} \otimes_{R_\chi,\varphi} A, X).$$

Thus

$$\boxed{\mathrm{Sel}(Ad(X)) \cong \mathrm{Hom}_A(\Omega_{R_\chi/B} \otimes_{R_\chi,\varphi} A, X).}$$

§**1.27. Theorem** $\mathrm{Sel}(Ad(\rho_A))^\vee \cong \Omega_{R_\chi/B} \otimes_{R_\chi,\varphi} A$.

Proof. Take the Pontryagin dual

$A^\vee := \mathrm{Hom}_B(A, B^\vee) = \mathrm{Hom}_{\mathbb{Z}_p}(A \otimes_B B, \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$.

Since $A = \varprojlim_i A_i$ for finite i and $\mathbb{Q}_p/\mathbb{Z}_p = \varinjlim_j p^{-1}\mathbb{Z}/\mathbb{Z}$, $A^\vee = \varinjlim_i \mathrm{Hom}(A_i, \mathbb{Q}_p/\mathbb{Z}_p) = \varinjlim_i A_i^\vee$ is a union of the finite modules $A_i^\vee$. We define $\mathrm{Sel}(Ad(\rho_A)) := \varinjlim_j \mathrm{Sel}(Ad(A_i^\vee))$. Defining $\Phi(A[A^\vee]) = \varinjlim_i \Phi(A[A_i^\vee])$, we see from compatibility of cohomology with injective limit

$$\Phi(A[A^\vee]) = \mathrm{Sel}(Ad(\rho_A)) = \varinjlim_i \mathrm{Sel}(Ad(A_i^\vee))$$

$$= \varinjlim_j \mathrm{Ker}(H^1(G, Ad(A_i^\vee)) \to \frac{H^1(D_p, Ad(A_i^\vee))}{F_-^+ H^1(D_p, Ad(A_i^\vee))})$$

§**1.28. Proof continues.** By the boxed formula in §1.25,

$$\mathrm{Sel}(Ad(\rho_A)) = \varinjlim_i \mathrm{Sel}(Ad(A_i^\vee)) = \varinjlim_i \mathrm{Hom}_{R_\chi}(\Omega_{R_\chi/B} \otimes_{R_\chi} A, A_i^\vee)$$
$$= \mathrm{Hom}_A(\Omega_{R_\chi/B} \otimes_{R_\chi} A, A^\vee) = \mathrm{Hom}_A(\Omega_{R_\chi/B} \otimes_{R_\chi} A, \mathrm{Hom}_{\mathbb{Z}_p}(A, \mathbb{Q}_p/\mathbb{Z}_p))$$
$$= \mathrm{Hom}_{\mathbb{Z}_p}(\Omega_{R_\chi/B} \otimes_{R_\chi} A, \mathbb{Q}_p/\mathbb{Z}_p) = (\Omega_{R_\chi/B} \otimes_{R_\chi} A)^\vee.$$

Taking Pontryagin dual back, we finally get

$$\boxed{\mathrm{Sel}(Ad(\rho_A))^\vee \cong \Omega_{R_\chi/B} \otimes_{R_{\chi,\varphi}} A \text{ and } \mathrm{Sel}(Ad(\overline{\rho}))^\vee \cong \Omega_{R_\chi/B} \otimes_{R_\chi} \mathbb{F}}$$

as desired. In particular, $\mathrm{Sel}(Ad(\boldsymbol{\rho}_\chi))^\vee = \Omega_{R_\chi/B}$ (with $\boldsymbol{\rho}_{\boldsymbol{\kappa}} = \boldsymbol{\rho}^{ord}$ if $\chi = \boldsymbol{\kappa}$). □

This is the generalization of the formula in §0.19

$$C_k \cong \Omega_{\mathbb{Z}_p[C_k]/\mathbb{Z}_p} \otimes_{\mathbb{Z}_p[C_k]} \mathbb{Z}_p.$$

§**1.29.** p**-Local condition.** The submodule $\Phi_p(A[X])$ in the cohomology group $H^1(\mathbb{Q}_p, Ad(X))$ is made of classes of 1-cocycles u with $u|_{I_p}$ is upper nilpotent and $u|_{D_p}$ is upper triangular with respect to the compatible basis (v_p, w_p). Suppose we have $\sigma \in I_p$ such that $\rho_A(\sigma) = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$ such that $\alpha \not\equiv \beta \mod \mathfrak{m}_A$. Suppose u is upper nilpotent over I_p. Then for $\tau \in D_p$, we have $Ad(\rho_A)(\tau)u(\tau^{-1}\sigma\tau) = (Ad(\rho_A)(\sigma)-1)u(\tau)+u(\sigma)$. Writing $u(\tau) = \left(\begin{smallmatrix}a & b\\ c & -a\end{smallmatrix}\right)$, we find $(Ad(\rho_A)(\sigma) - 1)u(\tau) = \left(\begin{smallmatrix}0 & (\alpha\beta^{-1}-1)b\\ (\alpha^{-1}\beta-1)c & 0\end{smallmatrix}\right)$. Since $\rho_A(\tau)$ is upper triangular and $u(\tau^{-1}\sigma\tau)$ is upper nilpotent, $Ad(\rho_A)(\tau)u(\tau^{-1}\sigma\tau)$ is still upper nilpotent; so, $(\alpha^{-1}\beta - 1)c = 0$ and hence $c = 0$. Therefore u is forced to be upper triangular over D_p. Thus we get

Lemma 4. *If $\overline{\rho}(\sigma)$ for at least one $\sigma \in I_p$ has two distinct eigenvalues, $\Phi_p(A[X])$ gives rise to the subgroup of $H^1(\mathbb{Q}_p, Ad(X))$ made of classes containing a 1-cocycle whose restriction to I_p is upper nilpotent.*