We construct the $p$-adic Riemann zeta function and Dirichlet $L$ function for each Dirichlet character modulo $p$-power by a method of N. Katz, interpolating Riemann zeta values and Dirichlet $L$-values at negative integers. The result can be extended to Dirichlet characters modulo $Np^n$ for any integer $N$ prime to $p$. Throughout, $W$ denote a complete $p$-adic discrete valuation ring containing $\mathbb{Z}_p$, whose field of fractions is denoted by $K$. 
§5.0. Slight modification of Euler’s theory.
Pick an integer $a > 1$. For an integer $n$, let $\xi(n) = 1$ if $a \nmid n$ and $\xi(n) = 1 - a$ if $a|n$. Let $\Psi(t) = \sum_{n=1}^{a} \xi(n) t^n$. If $a = 2$, we have $\Psi(t) = \frac{t}{1+t}$. Then we have a formal sum as Euler did:

$$(1 - a^{s+1}) \zeta(s) = \sum_{n} n^s - a \sum_{n} (an)^s = \sum_{n} \xi(n) n^s$$

and from this, we conclude for $g(t) = \sum_{n=1}^{\infty} \xi(n) t^n$ as Euler did:

$$\partial^m g(t) = \sum_{n} \xi(n) n^m t^n = (1 - a^{m+1}) \zeta(-m) \quad \text{for } \partial := t \frac{d}{dt}.$$ 

Note that $g(t) = \Psi(t)$. Thus after the same justification via the functional equation, we get, noting that $\sum_{n=1}^{a} \xi(n) t^n |_{t=1} = (a - 1) + (1 - a) = 0$ (so, the denominator of $\Psi$ does not have the factor $(t - 1)$ in it),

$$(1 - a^{m+1}) \zeta(-m) = \partial^m \Psi(t) |_{t=1} \quad \text{for all } m \in \mathbb{N}. \quad \text{(Ev)}$$

See a later half of [LFE, §2.1] for a full justification.
§5.1. Binomial differential operators.

Write \((x)_n = \sum_{m=0}^{n} c_{n,m} x^m \in \mathbb{Q}[x]\) and define \((\partial_n) = \sum_{m=0}^{n} c_{n,m} \partial^m\) as a differential operator.

**Lemma 1.** \((\partial_n) = \frac{t^n d^n}{n! dt^n}\) as a differential operator.

**Induction on** \(n\). Note \(\frac{x}{n+1} = \frac{(x-n)n!}{(n+1)!} (\frac{x}{n})\). Then we have

\[
\left(\frac{\partial}{n+1}\right)f = \frac{1}{(n+1)!} \left( t\frac{d}{dt} - n \right) t^n \frac{d^n f}{dt^n} = \frac{1}{(n+1)!} \left( nt^n \frac{d^n f}{dt^n} - nt^n \frac{d^n f}{dt^n} + t \frac{d^{n+1} f}{dt^{n+1}} \right)
\]

By Leibnitz, we have \(\frac{d^n fg}{dt^n} = \sum_{j=0}^{n} \binom{n}{j} \frac{d^j f}{dt^j} \cdot \frac{d^{n-j} g}{dt^{n-j}}\). So, by \(\binom{n}{j} = \frac{n!}{(n-j)!j!}\), we get

\[
\left(\frac{\partial}{n}\right)(fg) = \sum_{j=0}^{n} \left(\frac{\partial}{j}\right)f \cdot \left(\frac{\partial}{n-j}\right)g.
\]

\((*)\)
§5.2. \( p \)-Adic interpolation of \( \binom{\partial}{n} N(t)/D(t) \big|_{t=1} \) if \( D(1) \neq 0 \).

Let \( R := \{ \frac{N(t)}{D(t)} | N(t), D(t) \in W[t] \text{ and } |D(1)|_p = 1 \} \).

**Lemma 2.** The set \( R \) is a subring of \( K(t) \) stable under \( \binom{\partial}{n} \) for all \( n \), where \( K = \text{Frac}(W) \). Similarly \( W[t] \) is stable under \( \binom{\partial}{n} \).

**Proof of the ring property.** Plainly \( R \) is stable under the product. We see

\[
\frac{N}{D} \pm \frac{N'}{D'} = \frac{ND' \pm N'D}{DD'} \in R
\]

as \( |DD'(1)|_p = |D(1)|_p |D'(1)|_p = 1 \). Since \( \partial x^m = mx^m \), we have \( \binom{\partial}{m} x^m = \binom{m}{n} x^m \), and hence \( W[t] \) is stable under \( \binom{\partial}{n} \). \( \square \)

*Start of induction on \( n \).* Take \( n = 1 \). Then

\[
\binom{\partial}{1} \frac{N}{D} = \frac{\partial N)D - N \partial D}{D^2}
\]

which shows the result. \( \square \)
§5.3. Induction and a theorem. By (*), $0 = \left(\frac{\partial}{\partial n}(DD^{-1})\right) \iff \sum_{j=0}^{n} \binom{\partial}{j} D \cdot \left(\frac{\partial}{n-j}\right)(D^{-1})$; so, we get

$$\left(\frac{\partial}{n}\right)(D^{-1}) = -D^{-1} \sum_{j=1}^{n} \binom{\partial}{j} D \cdot \left(\frac{\partial}{n-j}\right)(D^{-1}).$$

By the induction hypothesis, $\left(\frac{\partial}{n-j}\right)(D^{-1}) \in R$ if $j \geq 1$, and hence we conclude $\left(\frac{\partial}{n}\right)(D^{-1}) \in R$. Again by (*), we have

$$\left(\frac{\partial}{n}\right)\frac{N}{D} \overset{(*)}{=} \left(\frac{\partial}{n}\right)\frac{N}{D} + \left(\frac{\partial}{n-1}\right)N\left(\frac{\partial}{1}\right)D^{-1} + \cdots + N\left(\frac{\partial}{n}\right)(D^{-1}).$$

Since $\left(\frac{\partial}{n}\right)D^{-1} \in R$, we conclude that $\left(\frac{\partial}{n}\right)\frac{N}{D} \in R$ as $R$ is a ring. \qed

**Theorem 1.** Let $1 < a \in \mathbb{N}$ and suppose $p \nmid a$. Then there exists a unique $p$-adic measure $\zeta_a$ on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p$ such that

$$\int x^m d\zeta_a = (1 - a^{m+1})\zeta(-m)$$

for all $m \in \mathbb{N}$. 
§ 5.4. Proof. Define $\int \binom{x}{n} d\zeta_a := \binom{\partial}{n} \Psi(1)$. Since $\Psi(t) = \frac{N}{D}$ with $D(t) = 1 + t + t^2 + \cdots + t^{a-1}$ and $N(t) \in \mathbb{Z}[t]$, we find $|D(1)|_p = |a|_p = 1$ as $p \nmid a$. Thus $\left| \int \binom{x}{n} d\zeta_a \right|_p = \left| \binom{\partial}{n} \Psi(1) \right|_p \leq 1$ by Lemma 2. So $\zeta_a$ is a bounded measure with values in $\mathbb{Z}_p$; i.e., $|\zeta_p|_p \leq 1$. Writing $x^m = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ and comparing highest degree coefficients, we find that the above is a finite sum $x^m = \sum_{n=0}^{m} a_n \binom{a}{n}$ with $a_n \in \mathbb{Z}_p \cap \mathbb{Q}$. Then by (Ev), we have

$$\int x^m d\zeta_a = \sum_{n=0}^{m} a_n \binom{\partial}{n} \Psi(1) = (\partial^m \Psi)(1) = (1 - a^{m+1}) \zeta(-m).$$

As we already remarked, this formula for every $m \in \mathbb{N}$ determines $\zeta_a$. Thus we obtain the desired measure. $\square$
§5.5. \( R \hookrightarrow W[[T]] \).

Pick \( F(t) = \frac{N(t)}{D(t)} \in R \). Then we can define \( \mu_F \in M(\mathbb{Z}_p, W) \) by

\[
\int \binom{x}{n} d\mu_F = \binom{x}{n} F(1).
\]

Then in the same manner of the proof of \( \zeta_a \in M(\mathbb{Z}_p, W) \), we find \( \mu_F \in M(\mathbb{Z}_p, W) \). The Taylor expansion of \( F \) at \( t = 1 \) is given by

\[
F(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n F}{dt^n}(1) T^n = \sum_{n=0}^{\infty} \left( \int \binom{x}{n} d\mu_F \right) T^n = \Phi_{\mu_F}(T)
\]

for \( T = t - 1 \). In this way, we can embed \( R \) into \( W[[T]] \).
5.6. Fourier inversion formula. Let $\mu_{p^n}$ be the group of $p^n$-th roots of unity in $\overline{\mathbb{Q}}_p$. Consider $k = K[\mu_{p^n}]$. For $x \in \mathbb{Z}_p$, we expand $x = \sum_{m=0}^{\infty} c_m p^m$ with $0 \leq c_m < p$ (the $p$-adic expansion of $x$). Let $[x]_n := \sum_{m=0}^{n-1} c_m p^m$. Then $x \equiv [x]_n \mod p^n$, and $\mu_{p^n} \times \mathbb{Z}/p^n\mathbb{Z} \ni (\zeta, x) \mapsto \zeta^x := \zeta[x]_n \in \mu_{p^n}$ is a perfect duality pairing. Recall the Fourier transform for $\phi : \mathbb{Z}/p^n\mathbb{Z} \to k$ and the inverse Fourier transform for $\psi : \mu_{p^n} \to K$ given by

$$\mathcal{F}(\phi)(\zeta) = \sum_{b \in \mathbb{Z}/p^n\mathbb{Z}} \phi(b) \zeta^{-b} \quad \text{and} \quad \mathcal{F}^*(\psi)(x) = p^{-n} \sum_{\zeta \in \mu_{p^n}} \psi(\zeta) \zeta^x.$$ 

We have the following orthogonality relation

$$\sum_{\zeta \in \mu_{p^n}} \zeta^{x-b} = \begin{cases} p^n & \text{if } x \equiv b \mod p^n\mathbb{Z}_p, \\ 0 & \text{otherwise}. \end{cases}$$

By this, $\phi = \mathcal{F}^* \mathcal{F}(\phi) = p^{-n} \sum_{b \in \mathbb{Z}/p^n\mathbb{Z}} \phi(b) \sum_{\zeta \in \mu_{p^n}} \zeta^{x-b}$. 

Exercise: Prove the above inversion formula.
§5.7. Multiplying a measure $\varphi \in M(\mathbb{Z}_p, W)$ by a function $f \in C(\mathbb{Z}_p, W)$. Define $f\varphi \in M(\mathbb{Z}_p, W)$ by $\int gd(f\varphi) = \int f \cdot gd\varphi$. Obviously $f\varphi \in M(\mathbb{Z}_p, W)$ (check this as an exercise).

**Lemma 3.** Fix $z \in 1 + m_W$. For $x \mapsto z^x = \sum_{n=0}^{\infty} \binom{x}{n}(z - 1)^n$, we have $\Phi_{z^x}\varphi(t) = \Phi\varphi(zt)$.

**Proof.** By definition,

$$\Phi_{z^x}\varphi(t) = \int z^x t^x d\varphi(x) = \int (zt)^x d\varphi = \Phi\varphi(zt)$$

as desired. $\square$

Regard $\phi : \mathbb{Z}/p^n\mathbb{Z} \to W$ as an element in $C(\mathbb{Z}_p, W[\mu_{p^n}])$ and define $[\phi]\Phi(t) := p^{-n} \sum_{b \in (\mathbb{Z}/p^n\mathbb{Z})} \phi(b) \sum_{\zeta \in \mu_{p^n}} \zeta^{-b}\Phi(\zeta t)$ for $\Phi \in W[[t]]$. A priori, $[\phi]\Phi \in k[[T]]$. 
§5.8. Multiplying a measure $\varphi$ by a locally constant function $\phi$.

**Corollary 1.** We have $[\varphi] \Phi \varphi = \Phi \varphi$. In particular, $[\varphi]$ is a $W$-linear operator acting on $W[[T]]$.

**Proof.** By Fourier inversion, we have

$$
\phi = \mathcal{F}^* \mathcal{F}(\phi) = p^{-n} \sum_{b \in \mathbb{Z}/p^n \mathbb{Z}} \phi(b) \sum_{\zeta \in \mu_{p^n}} \zeta^{x-b}.
$$

Thus $\int \phi f d\varphi = p^{-n} \sum_{b \in \mathbb{Z}/p^n \mathbb{Z}} \phi(b) \sum_{\zeta \in \mu_{p^n}} \zeta^{-b} \zeta^x f d\varphi$. This shows $\Phi \varphi = p^{-n} \sum_{b \in \mathbb{Z}/p^n \mathbb{Z}} \phi(b) \sum_{\zeta \in \mu_{p^n}} \zeta^{-b} \Phi \zeta^x \varphi$. Then by the lemma in §5.7 applied to $z = \zeta$, we get the desired formula. The last assertion follows from $M(\mathbb{Z}_p, W) \cong W[[T]]$ given by $\varphi \mapsto \Phi \varphi$. $\square$

- If $\Phi(\zeta t) = \Phi(t)$ for all $\zeta \in \mu_{p^n}$, then $[\phi](\Phi \Theta) = [\phi](\Theta) \Phi$,
- $[\phi] t^m = p^{-n} \sum_b \phi(b) \sum_\zeta \zeta^{-b} (\zeta t)^m = p^{-n} \sum_b \phi(b) t^m \sum_\zeta \zeta^{m-b} = \phi(m) t^m$,
- $[\phi] \partial \Phi = \partial([\phi] \Phi)$.
§5.9. Power series of the zeta measure.

Let $\Psi := \Phi_{\zeta_a}$ for $a \geq 2$ prime to $p$. By construction, we have

$$\Psi(t) = \frac{\sum_{b=1}^{a} \xi(b)t^b}{1-t^a} = \frac{t}{1-t} - \frac{a t^a}{1-t^a} = \partial \left( \log \frac{1-t^a}{1-t} \right).$$

By $\frac{1}{1-t^a} = 1 + t^a + t^{2a} + \cdots + t^{na} + \cdots$ (the geometric series),

$$\Psi(t) = \sum_{m=1}^{\infty} \xi(m)t^m = \sum_{b=1}^{ap^n} \xi(b)t^b \sum_{m=0}^{\infty} t^{ap^nm} = \frac{\sum_{b=1}^{ap^n} \xi(b)t^b}{1-t^{ap^n}}.$$

Write $\Theta := \sum_{b=1}^{ap^n} \xi(b)t^b$ and $\Phi := \frac{1}{1-t^{ap^n}}$ (in $W[[T]]$). Then $\Psi = \Phi\Theta$ and

$$[\phi]\Theta = \sum_{b=1}^{ap^n} \xi(b)\phi(b)t^b.$$

Since $\Phi(\zeta t) = \Phi(t)$ for $\zeta \in \mu_{p^n}$, we have

$$[\phi]\Psi = \Phi[\phi]\Theta = \sum_{m=1}^{\infty} \xi(m)\phi(m)t^m = \sum_{m=1}^{\infty} \phi(m)t^m - a \sum_{m=1}^{\infty} \phi(am)t^{am}.$$
5.10. Relation to complex L function.

Write $\phi_a(n) = \phi(n)$ if $a|n$ and $\phi_a(n) = 0$ otherwise. Suppose that
$\phi$ actually have values in $\overline{\mathbb{Q}}$. Since $\overline{\mathbb{Q}} \subset \mathbb{C}$, we may assume that
$\phi$ and $\phi_a$ are complex valued. So we have complex L function

$$L(s, \phi - a\phi_a) = \sum_{n=1}^{\infty} (\phi(n) - a\phi_a(n))n^{-s} \text{ (Re}(s) > 1).$$

Writing this as a linear combination of Hurwitz zeta function, it
has meromorphic continuation to the whole complex $s$-plane.
Now regarding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$, we have

$$L(-m, \phi - a^{m+1}\phi_a) = \partial^m([\phi]\Psi)|_{t=1} = \int \phi(x)x^md\zeta_a(x).$$

Take a Dirichlet character $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ and regard it as
a locally constant function on $\mathbb{Z}_p^\times$. Then extend it to $\mathbb{Z}_p$ by 0 outside $\mathbb{Z}_p^\times$. By $\chi_a(an) = \chi(an) = \chi(a)\chi(n)$, we have

$$L(-m, \chi - a^{m+1}\chi_a) = (1 - a^{m+1}\chi(a))(1 - \chi_0(p)p^m)L(-m, \chi_0),$$
where $\chi_0 = \chi$ if $\chi$ is non-trivial and $\chi_0 = 1$ all over $\mathbb{Z}_p$ if $\chi$ is
trivial.
§5.11. Evaluation formula.

**Theorem 2.** If $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ is a primitive character (when $\chi$ is trivial, we agree to put $n = 0$), then for all $m \in \mathbb{N}$,

$$\int \chi(x)x^md\zeta_a(x) = (1 - a^{m+1}\chi(a))(1 - \chi_0(p)p^m)L(-m, \chi_0).$$

If $\chi$ is even, for a primitive $p^n$-th root $\zeta$,

$$\int \chi(x)x^{-1}d\zeta_a(x) = \begin{cases} (1 - p^{-1}) \log(a) & \text{if } \chi = 1, \\ -(1 - \chi(a))p^{-n}G_\zeta(\chi) \sum_{b \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi^{-1}(b) \log(1 - \zeta^b) & \text{if } \chi \neq 1, \end{cases}$$

where $\log$ is the $p$-adic logarithm function such that $\log(x) = \sum_{n=1}^\infty (-1)^{n+1}(x-1)^n/n$ for $x$ with $|x|_p < 1$ and $G_\zeta(\chi) := \sum_{b=1}^{p^n} \chi(b)\zeta^b$ for a primitive root $\zeta \in \mu_{p^n}$.

Note that $L(-m, \chi) \neq 0$ if and only if $\chi(-1)(-1)^m = -1$; so, as a measure $d\zeta_a$ on $\mathbb{Z}_p^\times$ has non-trivial value only for even characters.
The first formula is proven before the theorem. We prove the second. For simplicity, we assume that \( a \equiv 1 \mod p \). We take care the general cases later. Since \( \frac{1-t^a}{1-t} \bigg|_{t=1} = a \) is in the domain of convergence of the \( p \)-adic logarithm \( \log \); so, we expand \( \log \left( \frac{1-t^a}{1-t} \right) \) into a power series in \( T = t - 1 \), and write it as \( \Phi \). Write this powerseries as \( \Phi(T) \). Then \( \Psi = \partial \Phi \) by the formula in §5.9. By the formula of §5.8, \( \partial \) and \( [\chi] \) commute, and we have \( [\chi]\Psi = \partial([\chi]\Phi) = (1 + T)\frac{d[\chi]\Phi}{dT} \). Let \( \varphi := x^{-1}\chi(x) \cdot \zeta_a(x) \in M(\mathbb{Z}_p, W) \). Since \( \Phi_{x\varphi} = \partial\Phi_{\varphi} \), we have \( \partial\Phi_{\varphi} = [\chi]\Psi \). Therefore \( \partial(\Phi_{\varphi} - [\chi]\Phi) = 0 \); so, \( \Phi_{\varphi} - [\chi]\Phi \in \mathbb{Q}_p \) (as the fundamental theorem of calculus is valid in \( \mathbb{Q}_p[[T]] \)). Since \( [\phi]t^m = \phi(m)t^m \), \([\chi]\) kills constants. This implies \( \Phi_{\varphi} = [1]\Phi_{\varphi} = [1][\chi]\Phi = [\chi]\Phi \). Therefore, we get

\[
\int x^{-1}\chi(x)d\zeta_a = [\chi]\Phi(1) = p^{-n} \sum_{b \in \mathbb{Z}/p^n\mathbb{Z}} \chi(b) \sum_{\zeta \in \mu_{pn}} \zeta^{-b}\Phi(\zeta)
\]

\[
= p^{-n} \sum_{\zeta \in \mu_{pn}} \Phi(\zeta) \sum_{b \in \mathbb{Z}/p^n\mathbb{Z}} \chi(b)\zeta^{-b} = p^{-n} \sum_{\zeta \in \mu_{pn}} G_\zeta(\chi)\Phi(\zeta).
\]
§5.13. Evaluation for primitive $\chi \neq 1$.
Since $\chi$ is primitive, $\chi|_{1+p^{n-1}\mathbb{Z}_p} \neq 1$. If $\zeta = \exp(-\frac{2\pi ic}{p^n})$, as we saw earlier, $G_\zeta(\chi) = \sum_{b \in \mathbb{Z}/p^n\mathbb{Z}} \chi(b)\zeta^{-b} = \chi^{-1}(c)G(\chi)$ for the standard $G(\chi)$ defined by $\exp(\frac{2\pi i}{p^n})$. Therefore

$$[\chi]\Phi(1) = p^{-n}G(\chi) \sum_{c \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi^{-1}(c)\Phi(\zeta^{-c})$$

for $\zeta = \exp(\frac{2\pi i}{p^n})$. Since $\Phi(\zeta^{-c}) = \log(1 - \zeta^{-ac}) - \log(1 - \zeta^{-c})$, by variable change, we get the factor $(1 - \chi(a))$ as desired.

If $\chi = 1$, we compute the sum over $b \in \mathbb{Z}/p\mathbb{Z}$, we have

$$G_\zeta(\chi) = \sum_{b \in \mathbb{Z}/p\mathbb{Z}} \chi(b) = \begin{cases} p - 1 & \text{if } \zeta = 1, \\ -1 & \text{if } \zeta \neq 1. \end{cases}$$

Since $a = 1 + mp$, $\zeta^a = \zeta$; so, $\frac{1-t^a}{1-t}|_{t=\zeta} = 1$ for $1 \neq \zeta \in \mu_p$ which is killed by log. Since $\frac{1-t^a}{1-t}|_{t=1} = a$, we get the desired formula. For general $a$, we will later see the formula. $\square$

Define $p = 4$ if $p = 2$ and $p = p$ if $p > 2$. Let $\Gamma := 1 + p\mathbb{Z}_p$. Then for $u = 1 + p$, we have $\Gamma = u^{\mathbb{Z}_p} := \{u^s | s \in \mathbb{Z}_p\}$, where $u^s = \sum_{n=0}^{\infty} \binom{s}{n} p^n$. Thus the multiplicative group $\Gamma$ is isomorphic to the additive group $\mathbb{Z}_p$. Then $\mathbb{Z}_p^\times = \mu \times \Gamma$ for $\mu = \mu_{p-1}$ if $p > 2$ and $\mu = \mu_2$ if $p = 2$. Write $\omega : \mathbb{Z}_p^\times \to \mu$ and $\langle \cdot \rangle : \mathbb{Z}_p^\times \to \Gamma$ for the projection. Define the $p$-adic L function for an even primitive character $\chi$ modulo $p^n$ by

$$L_p(s, \chi) := (1 - \chi(a)\langle a \rangle^{1-s})^{-1} \int_{\mathbb{Z}_p^\times} \chi\omega^{-1}\langle x \rangle^{-s} d\zeta_a(x).$$

By the evaluation formula in §5.11, we get

$$L_p(-m, \chi) = (1 - (\chi\omega^{m-1})_p(p)p^m))L(-m, (\chi\omega^{-m-1})_0) \text{ for all } m \in \mathbb{N}.$$ 

Here the left-hand-side is the $p$-adic value of the $p$-adic L and the right-hand-side is the complex value of the complex L. They are in the common subfield $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$; so, we have the identity.
§5.15. Analyticity. Define a $p$-adic measure $\zeta_{a,\chi} \in M(\Gamma, W)$ by
\[
\int_{\Gamma} \phi(\gamma) d\zeta_{a,\chi}(\gamma) := \int_{\mathbb{Z}_p^\times} \chi \omega^{-1}(x) \phi(\langle x \rangle) d\zeta_a.
\]
Identify $\mathbb{Z}_p \cong \Gamma$ by $\iota : \mathbb{Z}_p \ni x \mapsto u^x \in \Gamma$. For $\varphi = \iota^* \zeta_{a,\chi}$
\[
(1 - \chi(a) \langle a \rangle^{1-s}) L_p(s, \chi) = \int_{\Gamma} \gamma^{-s} d\zeta_{a,\chi} = \int_{\mathbb{Z}_p} u^{-sx} d\zeta_{a,\chi}(x) = \Phi_\varphi(u^{-s}).
\]
Define $\Phi_{a,\chi}(t) := \Phi_\varphi(u^{-1}t)$. Then
\[
L_p(1 - s, \chi^{-1}) = (1 - \chi^{-1}(a) \langle a \rangle^s)^{-1} \Phi_{a,\chi^{-1}}(u^s).
\]
Since $u^s = \exp(s \log(u))$ is an analytic function of $s$, $L_p(s, \chi)$ is a meromorphic function of $s$, whose pole comes from the zero of the denominator $(1 - \chi(a) \langle a \rangle^{1-s})$, that is $s = 1$ when $\chi(a) = 1$. Since $L_p(s, \chi)$ is independent of the choice of $a$, if $\chi \neq 1$, we can choose $a$ so that $\chi(a) \neq 1$. Therefore $L_p(s, \chi)$ is a holomorphic function at $s = 1$. If $\chi = 1$ it has a pole at $s = 1$ whose residue is $(1 - p^{-1})$ as $\langle a \rangle^s = 1 + \log(a)s + \text{higher terms}$. By the independence of $L_p$ from $a$, the evaluation formula in §5.11 is valid for all $a \geq 2$ prime to $p$. 
§5.16. **Theorem.** For each even primitive Dirichlet character $\chi$ modulo $p^n$, there exists a unique $p$-adic analytic function $L_p(s, \chi)$ on $\mathbb{Z}_p - \{1\}$ such that

$$L_p(-m, \chi) = (1-(\chi^m p)^{-1})_p(p^{m})L(-m,(\chi^{-m} p)^{-1})_0$$

for all $m \in \mathbb{N}$. When $\chi \neq 1$, $L_p(s, \chi)$ is analytic even at $s = 1$ and

$$L_p(1, \chi) = -p^{-n}G(\chi) \sum_{b \in (\mathbb{Z}/p^n \mathbb{Z})^\times} \chi^{-1}(b) \log_p(1 - \zeta^b)$$

for $\zeta = \exp_\infty(2\pi i/p^n)$ (with complex exponential $\exp_\infty$), where $\log_p$ is the $p$-adic logarithm. If $\chi = 1$, $L_p(s, \chi)$ has a simple pole at $s = 1$ with residue $(1 - p^{-1})$.

There is a classical result of Dirichlet–Kummer whose $p$-adic analogue is the above theorem:

**Theorem 3.** For a primitive character $\chi : (\mathbb{Z}/p^n \mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ with $\chi(-1) = 1$, we have for the real logarithm $\log_\infty$,

$$L(1, \chi) = -p^{-n}G(\chi) \sum_{a=1}^{p^n} \chi^{-1}(a) \log_\infty |1 - \zeta_{a}^{p^m}|.$$
§5.17. Some remarks.
Write \( \chi = \varepsilon^{-1} \omega^{-b} \) for \( \varepsilon \) factoring through \( \Gamma = u^Z_p \) with \( \varepsilon(u) = \zeta \in \mu_{p^{n-1}} \). Suppose \( \varepsilon \neq 1 \). Then for \( \varphi = i^* \zeta_{a, \omega^b} \),

\[
L_p(s, \varepsilon^{-1} \omega^{-b}) = \int_{\Gamma} \varepsilon^{-1}(\gamma) \gamma^{-s} d\zeta_{a, \omega^{-b}} = \int_{\mathbb{Z}_p} \zeta^{-x} u^{-sx} d\zeta_{a, \chi}(x) = \Phi_{\varphi}(\zeta^{-1} u^{-s}).
\]

Then we have

\[
L_p(1 - s, \chi^{-1}) = \Phi_{a, \omega^b}(\varepsilon(u) u^s).
\]

Thus we have \( \Phi_{a, \chi^{-1}}(t) = \Phi_{a, \omega^b}(\zeta t) \), we only need to know \( \Phi_{a, \omega^b} \) for \( b = 0, \ldots, p - 1 \).