Lecture note No.5 for Math 205a Fall 2019

p-Adic Riemann zeta functions.

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We construct the p-adic Riemann zeta function and Dirichlet L function for each Dirichlet character modulo p-power by a method of N. Katz, interpolating Riemann zeta values and Dirichlet L-values at negative integers. The result can be extended to Dirichlet characters modulo Np^n for any integer N prime to p. Throughout, W denote a complete p-adic discrete valuation ring containing $\mathbb{Z}_p$, whose field of fractions is denoted by K.

§5.0. Slight modification of Euler's theory.

Pick an integer $a > 1$. For an integer n, let $\xi(n) = 1$ if $a \nmid n$ and $\xi(n) = 1 - a$ if $a|n$. Let $\Psi(t) = \frac{\sum_{n=1}^{a} \xi(n)t^n}{1-t^a}$. If $a = 2$, we have $\Psi(t) = \frac{t}{1+t}$. Then we have a formal sum as Euler did:

$$(1 - a^{s+1})\zeta(s) = \sum_n n^s - a\sum_n (an)^s = \sum_n \xi(n)n^s$$

and from this, we conclude for $g(t) = \sum_{n=1}^{\infty} \xi(n)t^n$ as Euler did:

$$\partial^m g(t) = \sum_n \xi(n)n^m t^n = (1 - a^{m+1})\zeta(-m) \quad \text{for } \partial := t\frac{d}{dt}.$$

Note that $g(t) = \Psi(t)$. Thus after the same justification via the functional equation, we get, noting that $\sum_{n=1}^{a} \xi(n)t^n|_{t=1} = (a - 1) + (1 - a) = 0$ (so, the denominator of Ψ does not have the factor $(t - 1)$ in it),

$$(1 - a^{m+1})\zeta(-m) = \partial^m \Psi(t)|_{t=1} \quad \text{for all } m \in \mathbb{N}. \tag{Ev}$$

See a later half of [LFE, §2.1] for a full justification.

§5.1. Binomial differential operators.

Write $\binom{x}{n} = \sum_{m=0}^{n} c_{n,m} x^m \in \mathbb{Q}[x]$ and define $\binom{\partial}{n} = \sum_{m=0}^{m} c_{n,m} \partial^m$ as a differential operator.

Lemma 1. *$\binom{\partial}{n} = \frac{t^n}{n!}\frac{d^n}{dt^n}$ as a differential operator.*

Induction on n. Note $\frac{x}{n+1} = \frac{(x-n)n!}{(n+1)!}\binom{x}{n}$. Then we have

$$\binom{\partial}{n+1} f = \frac{1}{(n+1)!}(t\frac{t}{dt} - n)t^n \frac{d^n f}{dt^n} = \frac{1}{(n+1)!}(nt^n \frac{d^n f}{dt^n} - nt^n \frac{d^n f}{dt^n} + t\frac{d^{n+1} f}{dt^{n+1}})$$

$\square$

By Leibnitz, we have $\frac{d^n fg}{dt^n} = \sum_{j=0}^{n} \binom{n}{j} \frac{d^j f}{d^j t} \cdot \frac{d^{n-j} g}{dt^{n-j}}$. So, by $\binom{n}{j} = \frac{n!}{(n-j)!j!}$, we get

$$\binom{\partial}{n}(fg) = \sum_{j=0}^{n} \binom{\partial}{j} f \cdot \binom{\partial}{n-j} g. \qquad (*)$$

§5.2. p-Adic interpolation of $\binom{\partial}{n}N(t)/D(t)|_{t=1}$ if $D(1)\neq 0$.

Let $R := \{\frac{N(t)}{D(t)} | N(t), D(t) \in W[t]$ and $|D(1)|_p = 1\}$.

Lemma 2. *The set R is a subring of $K(t)$ stable under $\binom{\partial}{n}$ for all n, where $K = \mathrm{Frac}(W)$. Similarly $W[t]$ is stable under $\binom{\partial}{n}$.*

Proof of the ring property. Plainly R is stable under the product. We see

$$\frac{N}{D} \pm \frac{N'}{D'} = \frac{ND' \pm N'D}{DD'} \in R$$

as $|DD'(1)|_p = |D(1)|_p |D'(1)|_p = 1$. Since $\partial x^m = m x^m$, we have $\binom{\partial}{n} x^m = \binom{m}{n} x^m$, and hence $W[t]$ is stable under $\binom{\partial}{n}$. □

Start of induction on n. Take $n = 1$. Then

$$\binom{\partial}{1}\frac{N}{D} = \frac{(\partial N)D - N\partial D}{D^2}$$

which shows the result. □

§5.3. Induction and a theorem. By $(*)$, $0 = \binom{\partial}{n}(DD^{-1}) \overset{(*)}{=} \sum_{j=0}^{n} \binom{\partial}{j} D \cdot \binom{\partial}{n-j}(D^{-1})$; so, we get

$$\binom{\partial}{n}(D^{-1}) = -D^{-1} \sum_{j=1}^{n} \binom{\partial}{j} D \cdot \binom{\partial}{n-j}(D^{-1}).$$

By the induction hypothesis, $\binom{\partial}{n-j}(D^{-1}) \in R$ if $j \geq 1$, and hence we conclude $\binom{\partial}{n}(D^{-1}) \in R$. Again by $(*)$, we have

$$\binom{\partial}{n}\frac{N}{D} \overset{(*)}{=} \frac{\binom{\partial}{n}N}{D} + \binom{\partial}{n-1}N\binom{\partial}{1}D^{-1} + \cdots + N\binom{\partial}{n}(D^{-1}).$$

Since $\binom{\partial}{n}D^{-1} \in R$, we conclude that $\binom{\partial}{n}\frac{N}{D} \in R$ as R is a ring. $\square$

Theorem 1. *Let $1 < a \in \mathbb{N}$ and suppose $p \nmid a$. Then there exists a unique p-adic measure ζ_a on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p$ such that $\int x^m d\zeta_a = (1 - a^{m+1})\zeta(-m)$ for all $m \in \mathbb{N}$.*

§**5.4. Proof.** Define $\int \binom{x}{n} d\zeta_a := \binom{\partial}{n}\Psi(1)$. Since $\Psi(t) = \frac{N}{D}$ with $D(t) = 1 + t + t^2 + \cdots + t^{a-1}$ and $N(t) \in \mathbb{Z}[t]$, we find $|D(1)|_p = |a|_p = 1$ as $p \nmid a$. Thus $\left|\int \binom{x}{n} d\zeta_a\right|_p = \left|\binom{\partial}{n}\Psi(1)\right|_p \leq 1$ by Lemma 2. So ζ_a is a bounded measure with values in $\mathbb{Z}_p$; i.e., $|\zeta_p|_p \leq 1$. Writing $x^m = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ and comparing highest degree coefficients, we find that the above is a finite sum $x^m = \sum_{n=0}^{m} a_n \binom{x}{n}$ with $a_n \in \mathbb{Z}_p \cap \mathbb{Q}$. Then by (Ev), we have

$$\int x^m d\zeta_a = \sum_{n=0}^{m} a_n \binom{\partial}{n} \Psi(1) = (\partial^m \Psi)(1) = (1 - a^{m+1})\zeta(-m).$$

As we already remarked, this formula for every $m \in \mathbb{N}$ determines ζ_a. Thus we obtain the desired measure. $\square$

§**5.5.** $R \hookrightarrow W[[T]]$.

Pick $F(t) = \frac{N(t)}{D(t)} \in R$. Then we can define $\mu_F \in M(\mathbb{Z}_p, W)$ by

$$\int \binom{x}{n} d\mu_F = (\binom{\partial}{n} F)(1).$$

Then in the same manner of the proof of $\zeta_a \in M(\mathbb{Z}_p, W)$, we find $\mu_F \in M(\mathbb{Z}_p, W)$. The Taylor expansion of F at $t = 1$ is given by

$$F(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n F}{dt^n}(1) T^n = \sum_{n=0} \left(\int \binom{x}{n} d\mu_F \right) T^n = \Phi_{\mu_F}(T)$$

for $T = t - 1$. In this way, we can embed R into $W[[T]]$.

§**5.6. Fourier inversion formula.** Let μ_{p^n} be the group of p^n-th roots of unity in $\overline{\mathbb{Q}}_p$. Consider $k = K[\mu_{p^n}]$. For $x \in \mathbb{Z}_p$, we expand $x = \sum_{m=0}^{\infty} c_m p^m$ with $0 \leq c_m < p$ (the p-adic expansion of x). Let $[x]_n := \sum_{m=0}^{n-1} c_m p^m$. Then $x \equiv [x_n] \mod p^n$, and $\mu_{p^n} \times \mathbb{Z}/p^n\mathbb{Z} \ni (\zeta, x) \mapsto \zeta^x := \zeta^{[x]_n} \in \mu_{p^n}$ is a perfect duality pairing. Recall the Fourier transform for $\phi : \mathbb{Z}/p^n\mathbb{Z} \to k$ and the inverse Fourier transform for $\psi : \mu_{p^n} \to K$ given by

$$\mathcal{F}(\phi)(\zeta) = \sum_{b \in \mathbb{Z}/p^n\mathbb{Z}} \phi(b)\zeta^{-b} \text{ and } \mathcal{F}^*(\psi)(x) = p^{-n} \sum_{\zeta \in \mu_{p^n}} \psi(\zeta)\zeta^x.$$

We have the following orthogonality relation

$$\sum_{\zeta \in \mu_{p^n}} \zeta^{x-b} = \begin{cases} p^n & \text{if } x \equiv b \mod p^n\mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}$$

By this, $\phi = \mathcal{F}^*\mathcal{F}(\phi) = p^{-n} \sum_{b \in (\mathbb{Z}/p^n\mathbb{Z}} \phi(b) \sum_{\zeta \in \mu_{p^n}} \zeta^{x-b}$.

Exercise: Prove the above inversion formula.

§**5.7. Multiplying a measure $\varphi \in M(\mathbb{Z}_p, W)$ by a function $f \in C(\mathbb{Z}_p, W)$.** Define $f\varphi \in M(\mathbb{Z}_p, W)$ by $\int g d(f\varphi) = \int f \cdot g d\varphi$. Obviously $f\varphi \in M(\mathbb{Z}_p, W)$ (check this as an exercise).

Lemma 3. *Fix $z \in 1 + \mathfrak{m}_W$. For $x \mapsto z^x = \sum_{n=0}^{\infty} \binom{x}{n}(z-1)^n$, we have $\Phi_{z^x\varphi}(t) = \Phi_\varphi(zt)$.*

Proof. By definition,

$$\Phi_{z^x\varphi}(t) = \int z^x t^x d\varphi(x) = \int (zt)^x d\varphi = \Phi_\varphi(zt)$$

as desired. □

Regard $\phi : \mathbb{Z}/p^n\mathbb{Z} \to W$ as an element in $C(\mathbb{Z}_p, W[\mu_{p^n}])$ and define $[\phi]\Phi(t) := p^{-n} \sum_{b \in (\mathbb{Z}/p^n\mathbb{Z})} \phi(b) \sum_{\zeta \in \mu_{p^n}} \zeta^{-b} \Phi(\zeta t)$ for $\Phi \in W[[t]]$. A priori, $[\phi]\Phi \in k[[T]]$.

§5.8. Multiplying a measure φ by a locally constaint function ϕ.

Corollary 1. *We have $[\phi]\Phi_\varphi = \Phi_{\phi\varphi}$. In particular, $[\phi]$ is a W-linear operator acting on $W[[T]]$.*

Proof. By Fourier inversion, we have

$$\phi = \mathcal{F}^*\mathcal{F}(\phi) = p^{-n} \sum_{b \in (\mathbb{Z}/p^n\mathbb{Z}} \phi(b) \sum_{\zeta \in \mu_{p^n}} \zeta^{x-b}.$$

Thus $\int \phi f d\varphi = p^{-n} \sum_{b \in (\mathbb{Z}/p^n\mathbb{Z}} \phi(b) \sum_{\zeta \in \mu_{p^n}} \zeta^{-b}\zeta^x f d\varphi$. This shows $\Phi_{\phi\varphi} = p^{-n} \sum_{b \in (\mathbb{Z}/p^n\mathbb{Z})} \phi(b) \sum_{\zeta \in \mu_{p^n}} \zeta^{-b}\Phi_{\zeta^x\varphi}$. Then by the lemma in §5.7 applied to $z = \zeta$, we get the desired formula. The last assertion follows from $M(\mathbb{Z}_p, W) \cong W[[T]]$ given by $\varphi \mapsto \Phi_\varphi$. $\square$

- If $\Phi(\zeta t) = \Phi(t)$ for all $\zeta \in \mu_{p^n}$, then $[\phi](\Phi\Theta) = [\phi](\Theta)\Phi$,
- $[\phi]t^m = p^{-n} \sum_b \phi(b) \sum_\zeta \zeta^{-b}(\zeta t)^m = p^{-n} \sum_b \phi(b) t^m \sum_\zeta \zeta^{m-b} = \phi(m)t^m$,
- $[\phi]\partial\Phi = \partial([\phi]\Phi)$.

§5.9. Power series of the zeta measure.

Let $\Psi := \Phi_{\zeta_a}$ for $a \geq 2$ prime to p. By construction, we have

$$\Psi(t) = \frac{\sum_{b=1}^{a} \xi(b) t^b}{1 - t^a} = \frac{t}{1-t} - \frac{a t^a}{1 - t^a} = \partial \left(\log \frac{1 - t^a}{1 - t} \right).$$

By $\frac{1}{1-t^a} = 1 + t^a + t^{2a} + \cdots + t^{na} + \cdots$ (the geometric series),

$$\Psi(t) = \sum_{m=1}^{\infty} \xi(m) t^m = \sum_{b=1}^{ap^n} \xi(b) t^b \sum_{m=0}^{\infty} t^{ap^n m} = \frac{\sum_{b=1}^{ap^n} \xi(b) t^b}{1 - t^{ap^n}}.$$

Write $\Theta := \sum_{b=1}^{ap^n} \xi(b) t^b$ and $\Phi := \frac{1}{1-t^{ap^n}}$ (in $W[[T]]$). Then $\Psi = \Phi\Theta$ and

$$[\phi]\Theta = \sum_{b=1}^{ap^n} \xi(b) \phi(b) t^b.$$

Since $\Phi(\zeta t) = \Phi(t)$ for $\zeta \in \mu_{p^n}$, we have

$$[\phi]\Psi = \Phi[\phi]\Theta = \sum_{m=1}^{\infty} \xi(m)\phi(m) t^m = \sum_{m=1}^{\infty} \phi(m) t^m - a \sum_{m=1}^{\infty} \phi(am) t^{am}.$$

§5.10. Relation to complex L function.

Write $\phi_a(n) = \phi(n)$ if $a|n$ and $\phi_a(n) = 0$ otherwise. Suppose that ϕ actually have values in $\overline{\mathbb{Q}}$. Since $\overline{\mathbb{Q}} \subset \mathbb{C}$, we may assume that ϕ and ϕ_a are complex valued. So we have complex L function

$$L(s, \phi - a\phi_a) = \sum_{n=1}^{\infty} (\phi(n) - a\phi_a(n))n^{-s} \ (\mathrm{Re}(s) > 1).$$

Writing this as a linear combination of Hurwitz zeta function, it has meromorphic continuation to the whole complex s-plane. Now regarding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$, we have

$$L(-m, \phi - a^{m+1}\phi_a) = \partial^m([\phi]\Psi)|_{t=1} = \int \phi(x)x^m d\zeta_a(x).$$

Take a Dirichlet character $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ and regard it as a locally constant function on $\mathbb{Z}_p^\times$. Then extend it to $\mathbb{Z}_p$ by 0 outside $\mathbb{Z}_p^\times$. By $\chi_a(an) = \chi(an) = \chi(a)\chi(n)$, we have

$$L(-m, \chi - a^{m+1}\chi_a) = (1 - a^{m+1}\chi(a))(1 - \chi_0(p)p^m)L(-m, \chi_0),$$

where $\chi_0 = \chi$ if χ is non-trivial and $\chi_0 = 1$ all over $\mathbb{Z}_p$ if χ is trivial.

§5.11. Evaluation formula.

Theorem 2. *If $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ is a primitive character (when χ is trivial, we agree to put $n = 0$), then for all $m \in \mathbb{N}$,*

$$\int \chi(x)x^m d\zeta_a(x) = (1 - a^{m+1}\chi(a))(1 - \chi_0(p)p^m)L(-m, \chi_0).$$

If χ is even, for a primitive p^n-th root ζ,

$$\int \chi(x)x^{-1}d\zeta_a(x)$$

$$= \begin{cases} (1 - p^{-1})\log(a) & \text{if } \chi = 1, \\ -(1 - \chi(a))p^{-n}G_\zeta(\chi)\sum_{b\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\chi^{-1}(b)\log(1 - \zeta^b) & \text{if } \chi \neq 1, \end{cases}$$

where $\log$ is the p-adic logarithm function such that $\log(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$ for x with $|x|_p < 1$ and $G_\zeta(\chi) := \sum_{b=1}^{p^n}\chi(b)\zeta^b$ for a primitive root $\zeta \in \mu_{p^n}$.

Note that $L(-m, \chi) \neq 0$ if and only if $\chi(-1)(-1)^m = -1$; so, as a measure $d\zeta_a$ on $\mathbb{Z}_p^\times$ has non-trivial value only for even characters.

§5.12. Proof.

The first formula is proven before the theorem. We prove the second. For simplicity, we assume that $a \equiv 1 \mod p$. We take care the general cases later. Since $\frac{1-t^a}{1-t}|_{t=1} = a$ is in the domain of convergence of the p-adic logarithm log; so, we expand $\log(\frac{1-t^a}{1-t})$ into a power series in $T = t-1$, and write it as Φ. Write this powerseries as $\Phi(T)$. Then $\Psi = \partial\Phi$ by the formula in §5.9. By the formula of §5.8, ∂ and $[\chi]$ commute, and we have $[\chi]\Psi = \partial([\chi]\Phi) = (1+T)\frac{d[\chi]\Phi}{dT}$. Let $\varphi := x^{-1}\chi(x)\cdot\zeta_a(x) \in M(\mathbb{Z}_p, W)$. Since $\Phi_{x\varphi} = \partial\Phi_\varphi$, we have $\partial\Phi_\varphi = [\chi]\Psi$. Therefore $\partial(\Phi_\varphi - [\chi]\Phi) = 0$; so, $\Phi_\varphi - [\chi]\Phi \in \overline{\mathbb{Q}}_p$ (as the fundamental theorem of calculus is valid in $\overline{\mathbb{Q}}_p[[T]]$). Since $[\phi]t^m = \phi(m)t^m$, $[\chi]$ kills constants. This implies $\Phi_\varphi = [1]\Phi_\varphi = [1][\chi]\Phi = [\chi]\Phi$. Therefore, we get

$$\int x^{-1}\chi(x)d\zeta_a = [\chi]\Phi(1) = p^{-n}\sum_{b\in\mathbb{Z}/p^n\mathbb{Z}}\chi(b)\sum_{\zeta\in\mu_{p^n}}\zeta^{-b}\Phi(\zeta)$$
$$= p^{-n}\sum_{\zeta\in\mu_{p^n}}\Phi(\zeta)\sum_{b\in\mathbb{Z}/p^n\mathbb{Z}}\chi(b)\zeta^{-b} = p^{-n}\sum_{\zeta\in\mu_{p^n}}G_\zeta(\chi)\Phi(\zeta).$$

§5.13. Evaluation for primitive $\chi \neq 1$.

Since χ is primitive, $\chi|_{1+p^{n-1}\mathbb{Z}_p} \neq 1$. If $\zeta = \exp(-\frac{2\pi i c}{p^n})$, as we saw earlier, $G_\zeta(\chi) = \sum_{b\in\mathbb{Z}/p^n\mathbb{Z}} \chi(b)\zeta^{-b} = \chi^{-1}(c)G(\chi)$ for the standard $G(\chi)$ defined by $\exp(\frac{2\pi i}{p^n})$. Therefore

$$[\chi]\Phi(1) = p^{-n}G(\chi) \sum_{c\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi^{-1}(c)\Phi(\zeta^{-c})$$

for $\zeta = \exp(\frac{2\pi i}{p^n})$. Since $\Phi(\zeta^{-c}) = \log(1-\zeta^{-ac}) - \log(1-\zeta^{-c})$, by variable change, we get the factor $(1-\chi(a))$ as desired.

If $\chi = 1$, we compute the sum over $b \in \mathbb{Z}/p\mathbb{Z}$, we have

$$G_\zeta(\chi) = \sum_{b\in\mathbb{Z}/p\mathbb{Z}} \chi(b) = \begin{cases} p-1 & \text{if } \zeta = 1, \\ -1 & \text{if } \zeta \neq 1. \end{cases}$$

Since $a = 1 + mp$, $\zeta^a = \zeta$; so, $\frac{1-t^a}{1-t}|_{t=\zeta} = 1$ for $1 \neq \zeta \in \mu_p$ which is killed by log. Since $\frac{1-t^a}{1-t}|_{t=1} = a$, we get the desired formula. For general a, we will later see the formula. $\square$

§5.14. p-adic L functions.

Define $\mathbf{p} = 4$ if $p = 2$ and $\mathbf{p} = p$ if $p > 2$. Let $\Gamma := 1 + \mathbf{p}\mathbb{Z}_p$. Then for $u = 1 + \mathbf{p}$, we have $\Gamma = u^{\mathbb{Z}_p} := \{u^s | s \in \mathbb{Z}_p\}$, where $u^s = \sum_{n=0}^{\infty} \binom{s}{n} \mathbf{p}^n$. Thus the multiplicative group Γ is isomorphic to the additive group $\mathbb{Z}_p$. Then $\mathbb{Z}_p^\times = \mu \times \Gamma$ for $\mu = \mu_{p-1}$ if $p > 2$ and $\mu = \mu_2$ if $p = 2$. Write $\omega : \mathbb{Z}_p^\times \twoheadrightarrow \mu$ and $\langle \cdot \rangle : \mathbb{Z}_p^\times \twoheadrightarrow \Gamma$ for the projection. Define the p-adic L function for an even primitive character χ modulo p^n by

$$L_p(s, \chi) := (1 - \chi(a)\langle a \rangle^{1-s})^{-1} \int_{\mathbb{Z}_p^\times} \chi\omega^{-1} \langle x \rangle^{-s} d\zeta_a(x).$$

By the evaluation formula in §5.11, we get

$$L_p(-m, \chi) = (1 - (\chi\omega^{m-1})_p(p)p^m))L(-m, (\chi\omega^{-m-1})_0) \text{ for all } m \in \mathbb{N}.$$

Here the left-hand-side is the p-adic value of the p-adic L and the right-hand-side is the complex value of the complex L. They are in the common subfield $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$; so, we have the identity.

§5.15. Analyticity. Define a p-adic measure $\zeta_{a,\chi} \in M(\Gamma, W)$ by

$$\int_\Gamma \phi(\gamma) d\zeta_{a,\chi}(\gamma) := \int_{\mathbb{Z}_p^\times} \chi\omega^{-1}(x)\phi(\langle x\rangle) d\zeta_a.$$

Identify $\mathbb{Z}_p \cong \Gamma$ by $\iota : \mathbb{Z}_p \ni x \mapsto u^x \in \Gamma$. For $\varphi = \iota^*\zeta_{a,\chi}$

$$(1-\chi(a)\langle a\rangle^{1-s})L_p(s,\chi) = \int_\Gamma \gamma^{-s} d\zeta_{a,\chi} = \int_{\mathbb{Z}_p} u^{-sx} d\zeta_{a,\chi}(x) = \Phi_\varphi(u^{-s}).$$

Define $\Phi_{a,\chi}(t) := \Phi_\varphi(u^{-1}t)$. Then

$$L_p(1-s,\chi^{-1}) = (1-\chi^{-1}(a)\langle a\rangle^s)^{-1}\Phi_{a,\chi^{-1}}(u^s).$$

Since $u^s = \exp(s\log(u))$ is an analytic function of s, $L_p(s,\chi)$ is a meromorphic function of s, whose pole comes from the zero of the denominator $(1-\chi(a)\langle a\rangle^{1-s})$, that is $s=1$ when $\chi(a)=1$. Since $L_p(s,\chi)$ is independent of the choice of a, if $\chi \neq 1$, we can choose a so that $\chi(a) \neq 1$. Therefore $L_p(s,\chi)$ is a holomorphic function at $s=1$. If $\chi = 1$ it has a pole at $s=1$ whose residue is $(1-p^{-1})$ as $\langle a\rangle^s = 1+\log(a)s+$ higher terms. By the independence of L_p from a, the evaluation formula in §5.11 is valid for all $a \geq 2$ prime to p.

§**5.16. Theorem.** *For each even primitive Dirichlet character* χ *modulo* p^n, *there exists a unique* p*-adic analytic function* $L_p(s,\chi)$ *on* $\mathbb{Z}_p - \{1\}$ *such that*

$$L_p(-m,\chi) = (1-(\chi\omega^{m-1})_p(p)p^m))L(-m,(\chi\omega^{-m-1})_0) \quad \text{for all } m \in \mathbb{N}.$$

When $\chi \neq 1$, $L_p(s,\chi)$ *is analytic even at* $s = 1$ *and*

$$L_p(1,\chi) = -p^{-n}G(\chi) \sum_{b \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi^{-1}(b)\log_p(1-\zeta^b)$$

for $\zeta = \exp_\infty(2\pi i/p^n)$ *(with complex exponential* $\exp_\infty$*), where* $\log_p$ *is the* p*-adic logarithm. If* $\chi = 1$, $L_p(s,\chi)$ *has a simple pole at* $s = 1$ *with residue* $(1-p^{-1})$.

There is a classical result of Dirichlet–Kummer whose p-adic analogue is the above theorem:

Theorem 3. *For a primitive character* $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ *with* $\chi(-1) = 1$, *we have for the real logarithm* $\log_\infty$,

$$L(1,\chi) = -p^{-n}G(\chi)\sum_{a=1}^{p^n} \chi^{-1}(a)\log_\infty|1-\zeta_{p^m}^a|.$$

§5.17. Some remarks.
Write $\chi = \varepsilon^{-1}\omega^{-b}$ for ε factoring through $\Gamma = u^{\mathbb{Z}_p}$ with $\varepsilon(u) = \zeta \in \mu_{p^{n-1}}$. Suppose $\varepsilon \neq 1$. Then for $\varphi = \iota^* \zeta_{a,\omega^b}$,

$$\begin{aligned} L_p(s, \varepsilon^{-1}\omega^{-b}) = \int_\Gamma \varepsilon^{-1}(\gamma)\gamma^{-s} d\zeta_{a,\omega^{-b}} \\ = \int_{\mathbb{Z}_p} \zeta^{-x} u^{-sx} d\zeta_{a,\chi}(x) = \Phi_\varphi(\zeta^{-1}u^{-s}). \end{aligned}$$

Then we have

$$L_p(1-s, \chi^{-1}) = \Phi_{a,\omega^b}(\varepsilon(u)u^s).$$

Thus we have $\Phi_{a,\chi^{-1}}(t) = \Phi_{a,\omega^b}(\zeta t)$, we only need to know Φ_{a,ω^b} for $b = 0, \ldots, p-1$.