We prepare a measure theory for $p$-adic analytic interpolation of Dirichlet L-values. Let $A$ be a subring of a field containing $\mathbb{Q}$. We first show that any function $f : \mathbb{N} \rightarrow A$ can be written uniquely

$$f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n}$$

with

$$\binom{x}{n} = \begin{cases} \frac{x(x-1)\cdots(x-n+1)}{n!} & \text{if } n \geq 1, \\ 1 & \text{if } n = 0, \end{cases}$$

for $a_n(f) \in A$, where

$$a_n(f) = \sum_{i=0}^{n} (-1)^k \binom{n}{i} f(n-i) \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(i).$$

This is called an interpolation series expansion of $f$ and is a finite sum as $\binom{x}{n} = 0$ if $n > x$ (i.e., $f(x) = \sum_{n=0}^{x} a_n(f) \binom{x}{n}$).
§4.0. Binomial polynomial.
To prove (I), we list some properties of the binomial polynomial. If $0 < n \leq m \in \mathbb{Z}$, we have $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ is the binomial number; so, an integer, and if $n > m$, $\binom{m}{n} = 0$ as the factor $(x-m)$ shows up in the numerator. Thus the right-hand-side of (I) is a finite sum of $m + 1$ terms if $x = m$; so, the right-hand-side is well defined. Note that $\binom{x}{n}$ is an integer-valued polynomial over $\mathbb{N}$. We define a formal power series $(1+T)^x \in \mathbb{Q}[x][[T]]$ by

$$(1+T)^x := \sum_{n=0}^{\infty} \binom{x}{n} T^n.$$ 

For non-negative integer $x$, by the binomial theorem, we have $(1+T)^x \in \mathbb{Q}[T]$ gives the usual binomial expansion of degree $x$. We specialize $T$ to $-1$, then we have for $k \in \mathbb{N}$ and $x \in \mathbb{N} \cap [0,k]$,

$$(1-1)^{x-k} = \delta_{x,k} \quad (\text{D})$$

for the Kronecker symbol $\delta_{x,k}$. 

§4.1. Interpolation series. Note

\[
\binom{j+k}{k}(\begin{array}{c} x \\ j+k \end{array}) = \frac{(j+k)!x!}{k!j!(j+k)!(x-j-k)!} = \frac{x!(x-k)!}{k!(x-k)!j!(x-k-j)!} = \binom{x}{j}(x-k).
\]

Write the right-hand-side of (I) as \( f^* \) to prove \( f = f^* \). Then

\[
f^*(x) = \sum_{n=0}^{x} a_n(f)\binom{x}{n} = \sum_{n=0}^{x} \left[ \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k) \right] \binom{x}{n}
\]

\[
\overset{n-k\to j}{=} \sum_{k=0}^{x} f(k) \sum_{j=0}^{x-k} (-1)^{j} \binom{j+k}{j} \binom{x}{j+k}
\]

\[
= \sum_{k=0}^{x} f(k) \sum_{j=0}^{x-k} (-1)^{j} \binom{x}{j} \binom{x-k}{j}
\]

\[
= \sum_{k=0}^{x} f(k) \binom{x}{k} \sum_{j=0}^{x-k} (-1)^{j} \binom{x-k}{j} = \sum_{k=0}^{x} f(k) \binom{x}{k} \delta_{k,x} = f(x).
\]
4.2. Uniqueness of interpolation series expansion.

We prove $b_n = 0$ for all $n$ from an identity $\sum_{n=0}^{\infty} b_n(x_n) = 0$.

Towards a contradiction, we suppose $\{b_n\}_n \neq 0$ as a sequence. Pick the smallest integer $m$ such that $b_m \neq 0$. Then $\sum_{n=0}^{m-1} b_n(x_n) = 0$ for all integers $x$ and $\sum_{n=m+1}^{\infty} b_n(m_n) = 0$; so, $b_m = b_m(m_m) = \sum_{n=0}^{\infty} \binom{m}{n} = 0$, a contradiction. □

We have proven

**Proposition 1.** Let $A$ be an integral domain. For any function $f : \mathbb{N} \to A$, we have a unique interpolation series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n(f)(x_n),$$

where $a_n(f) = \sum_{i=0}^{n} (-1)^k \binom{n}{i} f(n-i) \overset{i \to n-i}{=} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(i)$.
§4.3. $p$-Adic interpolation series. Let $W$ be a discrete valuation ring containing $\mathbb{Z}_p$ with rank$_{\mathbb{Z}_p}W < \infty$. Write $| \cdot |_p$ for the absolute value of $W$ and extend it to the field of fractions $K := \text{Frac}(W)$. Let $A$ be a closed subring of $K$. For each function $f: \mathbb{Z}_p \to A$, we can restrict $f$ to $\mathbb{N} \subset \mathbb{Z}_p$ and define $a_n(f)$ as in the above proposition. Here is a theorem of Mahler

**Theorem 1** (K. Mahler, 1958). Let the notation be as above. For a function $f: \mathbb{Z}_p \to A$, we have

1. $a_n(f) \in A$;
2. $f$ is continuous if and only if $\lim_{n \to \infty} a_n(f) = 0$ under $| \cdot |_p$;
3. If $f$ is continuous, the interpolation series $\sum_{n=0}^{\infty} a_n(f) \binom{x}{n}$ converges to $f(x)$ for all $x \in \mathbb{Z}_p$. 
§4.4. Reduction to the case $A = W$.

Suppose $\lim_{n \to \infty} a_n(f) = 0$; so, $|a_n(f)|_p$ is bounded by $p^n$ for some $n \geq 0$. If $f$ is continuous, by compacity of $\mathbb{Z}_p$, $|f|_p$ is bounded by some $p^n$. Replacing $f$ by $p^n f$ for $n$ large, we may and do assume that $A = W$.

The series $\sum_{n=0}^{\infty} a_n(f) \binom{x}{n}$ converges uniformly; so, continuous, and the value on $\mathbb{N}$ coincide with $f$; so, by density of $\mathbb{N}$ in $\mathbb{Z}_p$, $f = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n}$ all over. Thus we need to prove one direction of (2):

$$\lim_{n \to \infty} a_n(f) = 0 \text{ if } f \text{ is continuous.}$$

We use valid interpolation expansion for $x \in \mathbb{N}$; so, hereafter $x, y \in \mathbb{N}$ until the end of the proof.
4.5. Topological ingredients and a distribution formula.

We list the topologically properties we use

- Any continuous functions \( f : D \to W \) on a compact set is uniformly continuous.
- The \( p \)-adic integer ring is compact (this follows from the embedding \( \mathbb{Z}_p \hookrightarrow \prod_n \mathbb{Z}/p^n\mathbb{Z} \) sending each \( p \)-adic expansion \( \sum_{n=0}^{\infty} a_n p^n \) to \( (\sum_{n=0}^{m} a_n p^n)_m \) is continuous and product of finite set is continuous under the product topology).
- \( x \mapsto (\binom{x}{n}) \) is a function from \( \mathbb{Z}_p \to \mathbb{Z}_p \). We know that \( \binom{x}{n} \) is continuous as it is a polynomial and has values in \( \mathbb{N} \) on \( \mathbb{N} \). Then this fact follows from the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{N} & \rightarrow & \mathbb{Z}_p \\
\downarrow \text{dense} & & \downarrow \text{dense} \\
\binom{x}{n} & \rightarrow & \binom{x}{n}
\end{array}
\]

Take \( y \in \mathbb{N} \). By comparing the coefficients of \( (1 + T)^x(1 + T)^y = (1 + T)^{x+y} \), we get a "\( x \) and \( y \)" distribution formula

\[
(\binom{x}{m}) = \sum_{n=0}^{m} \binom{x}{n} \binom{y}{m-n}.
\]
4.6. Summation interchange. Let $f_y(x) := f(x + y)$. Then $a_n(f_y) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k + y)$. Then the sum of interpolation series runs over $m \in [0, x + y]$, $n \in [0, x]$ and $m - n \in [0, y]$:

$$\sum_{n=0}^{\infty} \left[ a_n(f_y) \right] \left( \frac{x}{n} \right) = f_y(x) = f(x + y) = \sum_{m=0}^{x+y} a_m(f) \left( \frac{x + y}{m} \right)$$

(D)  $$\sum_{m=0}^{x+y} a_m(f) \sum_{n=0}^{m} \left( \binom{x}{n} \binom{y}{m-n} \right) \overset{(*)}{=} \sum_{m=0}^{\infty} a_m(f) \sum_{n=0}^{\infty} \left( \binom{x}{n} \binom{y}{m-n} \right)$$

(**)  $$\sum_{n=0}^{\infty} \left( \binom{x}{n} \sum_{m=n}^{\infty} a_m(f) \binom{y}{m-n} \right) \overset{m-n\rightarrow k}{=} \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{n+k}(f) \binom{y}{k} \right) \left( \binom{x}{n} \right).$$

(C)

Here at (*), we may think $n$ runs from 0 to $\infty$ as we have $\binom{x}{n} = 0$ if $n > x$, and similarly, we can run $m$ to $\infty$ as $x + y \geq y$ and $\binom{y}{m-n} = 0$ if $m - n \geq y$. We can interchange the sum at (**) as they are anyway finite summations and the sum for $m$ starts with $m = n$ (by $m - n \in [0, y]$).
4.7. Manipulation of coefficients.

Comparing the coefficient of \( \binom{x}{n} \) of the two boxed sides of (C), we get

\[
\sum_{k=0}^{\infty} a_{n+k}(f) \binom{y}{k} = \boxed{a_n(fy)} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k + y)
\]

Let \( y = p^t \). Then we bring the term for \( k < y = p^t \) to the RHS (right-hand-side) and get (the boxed terms are equal)

\[
 a_{n+p^t}(f) = -a_n(f) - \sum_{k=1}^{p^t-1} a_{n+k}(f) \binom{p^t}{k} + \boxed{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k + p^t)}.
\]

Since \( a_n(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k) \), the above formula produces

\[
 a_{n+p^t}(f) = - \sum_{k=1}^{p^t-1} a_{n+k}(f) \binom{p^t}{k} + \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (f(k + p^t) - f(k)).
\]
§4.8. Use of uniform continuity. Suppose $f$ is continuous. Compactness of $\mathbb{Z}_p$ tells us that $f$ is uniformly continuous: for any given $\varepsilon = p^{-s}$ ($s$ large), we have $t = t(s) > 0$ such that $|(k + p^t) - k|_p \leq p^{-t(s)} \Rightarrow |f(k + p^t) - f(k)|_p < p^{-s}$. Since $\left|\frac{p^t}{k}\right|_p \leq p^{-1}$ and $|\cdot|_p$ is non-Archimedean, we have

$$|a_{n+p^t}(f)|_p \leq \max(p^{-1}|a_{n+1}(f)|_p, \ldots, p^{-1}|a_{n+p^t-1}(f)|_p, p^{-s}).$$

Since $|a_n(f)|_p \leq 1$, we find if $n \geq p^{t(1)}$ $|a_n(f)|_p \leq p^{-1}$. Then $|a_{n+p^{t(2)}(f)}| \leq p^{-2}$ or equivalently, $|a_n(f)|_p \leq p^{-2}$ if $n \geq p^{t(1)} + p^{t(2)}$, and repeating this process, inductively we get

$$|a_n(f)|_p \leq p^{-m} \text{ if } n \geq p^{t(1)} + p^{t(2)} + \cdots + p^{t(m)}.$$  

Thus we get the desired limit formula: $\lim_{n \to \infty} a_n(f) = 0$. □

Exercise: Suppose $A = \mathbb{Q}_p$ and $f$ is continuous. Use compacity of $\mathbb{Z}_p$, prove that there exists $0 < \alpha \in \mathbb{Z}$ such that $p^\alpha f$ has values in $W$, and using this fact, prove $\lim_{n \to \infty} a_n(f) = 0$. 


§4.9. Space of continuous functions \(f : \mathbb{Z}_p \to A\).

Let \(C(\mathbb{Z}_p, A)\) be the space of all continuous functions \(f : \mathbb{Z}_p \to A\). Then \(f\) has unique interpolation series expansion

\[ f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n} \]

with \(a_n(f) \in A\). Define the norm of \(f\) by \(|f|_p := \sup_{x \in \mathbb{Z}_p} |f(x)|_p\). Plainly \(|\cdot|_p\) is a norm satisfying \(|f + g|_p \leq \max(|f|_p, |g|_p), |\alpha f|_p = |\alpha|_p |f|_p \) (\(\alpha \in A\)) and \(|f|_p = 0 \iff f = 0\).

**Corollary 1.** We have \(|f|_p = \sup_n |a_n(f)|_p\)

**Proof.** Note \(|a_n(f)|_p = |\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k)|_p \leq \max_k |f(k)|_p \leq |f|_p\), which shows \(\sup_n |a_n(f)|_p \leq |f|_p\). On the other hand, we have

\[ |f|_p = \lim_{N \to \infty} |\sum_{n=0}^{N} a_n(f) \binom{x}{n}|_p = \lim_{N \to \infty} |\sum_{n=0}^{N} a_n(f) \binom{x}{n}|_p \leq \sup_n |a_n(f)|_p \]

as desired. \(\square\)
§4.10. Space of $p$-adic measure on $\mathbb{Z}_p$.

Since uniform convergence preserves continuity, the space $C(\mathbb{Z}_p, A)$ is a Banach $A$-module (i.e., complete under $|\cdot|_p$). We write $M(\mathbb{Z}_p, A)$ for the space of $A$-linear functional $\varphi : C(\mathbb{Z}_p, A) \to A$ such that $|\varphi(f)|_p \leq B|f|_p$ for a constant $B > 0$. We often write $\int_{\mathbb{Z}_p} f d\varphi := \varphi(f)$. If $A = W$, then $B$ can be taken to be 1. We define

$$|\varphi|_p = \sup_{0 \neq f \in C(\mathbb{Z}_p, A)} \frac{|\varphi(f)|_p}{|f|_p} \overset{(*)}{=} \sup_{f \in C(\mathbb{Z}_p, A), |f|_p = 1} |\varphi(f)|_p.$$  

Exercise: Why the identity ($\ast$) holds? Prove $|\varphi + \varphi'|_p \leq \max(|\varphi|_p, |\varphi'|_p)$, $|\alpha \varphi|_p = |\alpha|_p |\varphi|_p$ for $\alpha \in A$ and $|\varphi|_p = 0 \Leftrightarrow \varphi = 0$.

If $\lim_{n \to \infty} \varphi_n = \varphi$ under the above norm, we have $|\varphi|_p = \lim_{n \to \infty} |\varphi_n|$ and $|\varphi_m(f) - \varphi_n(f)|_p \leq |\varphi_m - \varphi_n|_p |f|_p$, and hence $\{\varphi_n(f)\}_n$ is a Cauchy sequence. Thus we define $\varphi(f) := \lim_{n \to \infty} \varphi_n(f)$. Therefore $|\varphi(f)|_p = \lim_{n \to \infty} |\varphi_n(f)|_p \leq \lim_{n \to \infty} |\varphi_n|_p |f|_p = |\varphi|_p |f|_p$. Thus $\varphi \in M(\mathbb{Z}_p, A)$, and $M(\mathbb{Z}_p, A)$ is a Banach $A$-module.
4.11. Moment determines a measure.

Theorem 2. For a given sequence \( \{b_n \in A\}_n \) with bounded norm, there is a unique measure \( \varphi \in M(\mathbb{Z}_p, A) \) satisfying \( \int \left( \frac{x}{n} \right) d\varphi = b_n \) and \( |\varphi|_p = \sup_n |b_n|_p \) such that if \( f = \sum_{n=0}^{\infty} a_n(f) \left( \frac{x}{n} \right) \in C(\mathbb{Z}_p, A) \), we have \( \int f d\varphi = \sum_{n=0}^{\infty} b_n a_n(f) \). All element \( \varphi \) in \( M(\mathbb{Z}_p, A) \) is obtained this way.

Proof. Since \( \lim_{n \to \infty} a_n(f) = 0 \) and \( |b_n| \leq B \) for all \( n \), \( \sum_{n=0}^{\infty} b_n a_n(f) \) converges, giving an \( A \)-linear map \( \varphi : C(\mathbb{Z}_p, A) \to A \). Note

\[
| \sum_{n=0}^{\infty} b_n a_n(f) | = \sup_n |b_n|_p |a_n(f)|_p \leq B \sum_n |a_n(f)|_p = B |f|_p.
\]

So \( \varphi \in M(\mathbb{Z}_p, A) \). Taking \( B := \sup_n |b|_p \), we find \( |\varphi|_p \leq B \). Since \( | \int \left( \frac{x}{n} \right) d\varphi|_p = |b_n|_p \) and \( \left| \left( \frac{x}{n} \right) \right|_p = 1 \), we find \( B = |\varphi|_p \). Since \( f \) has unique interpolation series expansion, every measure is given by the above way. \( \square \)
§ 4.12. Corollary. \( \{ \int x^n d\varphi \}_n \) determines \( \varphi \in M(\mathbb{Z}_p, A) \).

By definition, we have \( M(\mathbb{Z}_p, A) \subset M(\mathbb{Z}_p, K) \). Since \( \binom{x}{n} = \sum_{j=0}^{n} a_j x^j \), the sequence \( \{ \int x^n d\varphi \}_n \) determines \( \int \binom{x}{n} d\varphi = \sum_{j=0}^{n} a_j \int x^j d\varphi \) as an element of \( K \). Then the above theorem implies \( \varphi \) is uniquely determined in \( M(\mathbb{Z}_p, K) \). Since \( M(\mathbb{Z}_p, A) \) is a subspace of \( M(\mathbb{Z}_p, K) \), we find \( \varphi \) is determined by \( \{ \int x^n d\varphi \}_n \in A^\mathbb{N} \).

Consider a formal expansion \( (1 + T)^x = \sum_{n=0}^\infty \binom{x}{n} T^n \). For each \( \varphi \in M(\mathbb{Z}_p, W) \), we define

\[
\Phi_\varphi(T) := \int (1 + T)^x d\varphi = \sum_{n=0}^\infty \int \binom{x}{n} d\varphi T^n \in W[[T]].
\]

By the above facts, this gives an isomorphism

\[
\Phi : M(\mathbb{Z}_p, W) \cong W[[T]].
\]
§4.13. Ring structure of $M(\mathbb{Z}_p, A)$.

For two measures $\varphi, \psi \in M(\mathbb{Z}_p, A)$, we define $\int f d(\varphi * \psi) := \int \int f(x + y) d\varphi(x) d\psi(y)$. Then

$$\Phi_{\varphi * \psi} = \int \int (1 + T)^{x+y} d\varphi(x) d\psi(y)$$

$$= \int (1 + T)^x d\varphi(x) \cdot \int (1 + T)^y d\psi(y) = \Phi_\varphi(T) \Phi_\psi(T).$$

Thus $M(\mathbb{Z}_p, W)$ with convolution product $(\varphi, \psi) \mapsto \varphi * \psi$ is isomorphic to the power series ring $W[[T]]$. Defining $|\sum_{n=0}^\infty a_n T^n| = \sum_n |a_n|_p$, $W[[T]]$ is a Banach $W$-module. The theorem in §4.10 tells us

**Corollary 2.** The isomorphism $\Phi : M(\mathbb{Z}_p, W) \cong W[[T]]$ is an isometry of normed rings.

Exercise: What is the multiplicative identity of the ring $M(\mathbb{Z}_p, W)$?