Lecture note No.4 for Math 205a Fall 2019

p-Adic integration under a p-adic measure.

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We prepare a measure theory for p-adic analytic interpolation of Dirichlet L-values. Let A be a subring of a field containing $\mathbb{Q}$. We first show that any function $f : \mathbb{N} \to A$ can be written uniquely

$$f(x) = \sum_{n=0}^{\infty} a_n(f)\binom{x}{n} \text{ with } \binom{x}{n} = \begin{cases} \frac{x(x-1)\cdots(x-n+1)}{n!} & \text{if } n \geq 1, \\ 1 & \text{if } n = 0, \end{cases} \tag{I}$$

for $a_n(f) \in A$, where

$$a_n(f) = \sum_{i=0}^{n} (-1)^k \binom{n}{i} f(n-i) \overset{i \mapsto n-i}{=} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(i).$$

This is called an *interpolation series expansion* of f and is a finite sum as $\binom{x}{n} = 0$ if $n > x$ (i.e., $f(x) = \sum_{n=0}^{x} a_n(f)\binom{x}{n}$).

§4.0. Binomial polynomial.

To prove (I), we list some propertyies of the binomial polynomial. If $0 < n \leq m \in \mathbb{Z}$, we have $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ is the binomial number; so, an integer, and if $n > m$, $\binom{m}{n} = 0$ as the factor $(x-m)$ shows up in the numerator. Thus the right-hand-side of (I) is a finite sum of $m+1$ terms if $x = m$; so, the right-hand-side is well defined. Note that $\binom{x}{n}$ is an integer-valued polynomial over $\mathbb{N}$. We define a formal power series $(1+T)^x \in \mathbb{Q}[x][[T]]$ by

$$(1+T)^x := \sum_{n=0}^{\infty} \binom{x}{n} T^n.$$

For non-negative integer x, by the binomial theorem, we have $(1+T)^x \in \mathbb{Q}[T]$ gives the usual binomial expansion of degree x. We specialize T to -1, then we have for $k \in \mathbb{N}$ and $x \in \mathbb{N} \cap [0,k]$,

$$\sum_{j=0}^{x-k} (-1)^j \binom{x-k}{j} = (1-1)^{x-k} = \delta_{x,k} \tag{D}$$

for the Kronecker symbol $\delta_{x,k}$.

§**4.1. Interpolation series.** Note

$$\binom{j+k}{k}\binom{x}{j+k} = \frac{(j+k)!x!}{k!j!(j+k)!(x-j-k)!}$$
$$= \frac{x!(x-k)!}{k!(x-k)!j!(x-k-j)!} = \binom{x}{k}\binom{x-k}{j}.$$

Write the right-hand-side of (I) as f^* to prove $f = f^*$. Then

$$f^*(x) = \sum_{n=0}^{x} a_n(f)\binom{x}{n} = \sum_{n=0}^{x}\left[\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(k)\right]\binom{x}{n}$$
$$\overset{n-k\mapsto j}{=} \sum_{k=0}^{x} f(k)\sum_{j=0}^{x-k}(-1)^j\binom{j+k}{k}\binom{x}{j+k}$$
$$= \sum_{k=0}^{x} f(k)\sum_{j=0}^{x-k}(-1)^j\binom{x}{k}\binom{x-k}{j}$$
$$= \sum_{k=0}^{x} f(k)\binom{x}{k}\sum_{j=0}^{x-k}(-1)^j\binom{x-k}{j} = \sum_{k=0}^{x} f(k)\binom{x}{k}\delta_{k,x} = f(x).$$

§4.2. Uniqueness of interpolation series expansion.

We prove $b_n = 0$ for all n from an identity $\sum_{n=0}^{\infty} b_n \binom{x}{n} = 0$.

Towards a contradiction, we suppose $\{b_n\}_n \neq 0$ as a sequence. Pick the smallest integer m such that $b_m \neq 0$. Then $\sum_{n=0}^{m-1} b_n \binom{x}{n} = 0$ for all integers x and $\sum_{n=m+1}^{\infty} b_n \binom{m}{n} = 0$; so, $b_m = b_m \binom{m}{m} = \sum_{n=0}^{\infty} \binom{m}{n} = 0$, a contradiction. $\square$

We have proven

Proposition 1. *Let A be an integral domain. For any function $f : \mathbb{N} \to A$, we have a unique interpolation series expansion*

$$f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n},$$

where $a_n(f) = \sum_{i=0}^{n} (-1)^k \binom{n}{i} f(n-i) \overset{i \mapsto n-i}{=} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(i)$.

§4.3. p-Adic interpolation series. Let W be a discrete valuation ring containing $\mathbb{Z}_p$ with $\mathrm{rank}_{\mathbb{Z}_p} W < \infty$. Write $|\cdot|_p$ for the absolute value of W and extend it to the field of fractions $K := \mathrm{Frac}(W)$. Let A be a closed subring of K. For each function $f : \mathbb{Z}_p \to A$, we can restrict f to $\mathbb{N} \subset \mathbb{Z}_p$ and define $a_n(f)$ as in the above proposition. Here is a theorem of Mahler

Theorem 1 (K. Mahler, 1958)**.** *Let the notation be as above. For a function $f : \mathbb{Z}_p \to A$, we have*
(1) $a_n(f) \in A$;
(2) f is continuous if and only if $\lim_{n\to\infty} a_n(f) = 0$ under $|\cdot|_p$;
(3) If f is continuous, the interpolation series $\sum_{n=0}^{\infty} a_n(f)\binom{x}{n}$ converges to $f(x)$ for all $x \in \mathbb{Z}_p$.

§4.4. Reduction to the case $A = W$.

Suppose $\lim_{n\to\infty} a_n(f) = 0$; so, $|a_n(f)|_p$ is bounded by p^n for some $n \geq 0$. If f is continuous, by compacity of $\mathbb{Z}_p$, $|f|_p$ is bounded by some p^n. Replacing f by $p^n f$ for n large, we may and do assume that $A = W$.

The series $\sum_{n=0}^{\infty} a_n(f)\binom{x}{n}$ converges uniformly; so, continuous, and the value on $\mathbb{N}$ coincide with f; so, by density of $\mathbb{N}$ in $\mathbb{Z}_p$, $f = \sum_{n=0}^{\infty} a_n(f)\binom{x}{n}$ all over. Thus we need to prove one direction of (2):

$$\lim_{n\to\infty} a_n(f) = 0 \text{ if } f \text{ is continuous.}$$

We use valid interpolation expansion for $x \in \mathbb{N}$; so, hereafter $x, y \in \mathbb{N}$ until the end of the proof.

§4.5. Topological ingredients and a distribution formula.

We list the topologically properties we use

- Any continuous functions $f : D \to W$ on a compact set is uniformly continuous.
- The p-adic integer ring is compact (this follows from the embedding $\mathbb{Z}_p \hookrightarrow \prod_n \mathbb{Z}/p^n\mathbb{Z}$ sending each p-adic expansion $\sum_{n=0}^{\infty} a_n p^n$ to $(\sum_{n=0}^{m} a_n p^n)_m$ is continuous and product of finite set is continuous under the product topology).
- $x \mapsto \binom{x}{n}$ is a function from $\mathbb{Z}_p \to \mathbb{Z}_p$. We know that $\binom{x}{n}$ is continuous as it is a polynomial and has values in $\mathbb{N}$ on $\mathbb{N}$. Then this fact follows from the following commutative diagram:

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow[\text{dense}]{\subset} & \mathbb{Z}_p \\ {\scriptstyle \binom{x}{n}}\Big\downarrow & & \Big\downarrow{\scriptstyle \binom{x}{n}} \\ \mathbb{N} & \xrightarrow[\text{dense}]{\subset} & \mathbb{Z}_p \end{array}$$

Take $y \in \mathbb{N}$. By comparing the coefficients of $(1+T)^x(1+T)^y = (1+T)^{x+y}$, we get a "$x$ and y" distribution formula

(D) $$\binom{x+y}{m} = \sum_{n=0}^{m} \binom{x}{n}\binom{y}{m-n}.$$

§**4.6. Summation interchange.** Let $f_y(x) := f(x+y)$. Then $a_n(f_y) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(k+y)$. Then the sum of interpolation series runs over $m \in [0, x+y]$, $n \in [0, x]$ and $m-n \in [0, y]$:

$$\sum_{n=0}^{\infty} \boxed{a_n(f_y)} \binom{x}{n} = f_y(x) = f(x+y) = \sum_{m=0}^{x+y} a_m(f) \binom{x+y}{m}$$
$$\stackrel{(D)}{=} \sum_{m=0}^{x+y} a_m(f) \sum_{n=0}^{m} \binom{x}{n}\binom{y}{m-n} \stackrel{(*)}{=} \sum_{m=0}^{\infty} a_m(f) \sum_{n=0}^{\infty} \binom{x}{n}\binom{y}{m-n}$$
$$\stackrel{(**)}{=} \sum_{n=0}^{\infty} \binom{x}{n} \sum_{m=n}^{\infty} a_m(f) \binom{y}{m-n} \stackrel{m-n\mapsto k}{=} \sum_{n=0}^{\infty} \boxed{\left(\sum_{k=0}^{\infty} a_{n+k}(f)\binom{y}{k}\right)} \binom{x}{n}. \tag{C}$$

Here at $(*)$, we may think n runs from 0 to ∞ as we have $\binom{x}{n} = 0$ if $n > x$, and similarly, we can run m to ∞ as $x + y \geq y$ and $\binom{y}{m-n} = 0$ if $m - n \geq y$. We can interchange the sum at $(**)$ as they are anyway finite summations and the sum for m starts with $m = n$ (by $m - n \in [0, y]$).

§4.7. Manipulation of coefficients.

Comparing the coefficient of $\binom{x}{n}$ of the two boxed sides of (C), we get

$$\sum_{k=0}^{\infty} a_{n+k}(f)\binom{y}{k} = \boxed{a_n(f_y)} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(k+y)$$

Let $y = p^t$. Then we bring the term for $k < y = p^t$ to the RHS (right-hand-side) and get (the boxed terms are equal)

$$a_{n+p^t}(f) = -a_n(f) - \sum_{k=1}^{p^t-1} a_{n+k}(f)\binom{p^t}{k} + \boxed{\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(k+p^t)}.$$

Since $a_n(f) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(k)$, the above formula produces

$$a_{n+p^t}(f) = -\sum_{k=1}^{p^t-1} a_{n+k}(f)\binom{p^t}{k} + \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(f(k+p^t)-f(k)).$$

§**4.8. Use of uniform continuity.** Suppose f is continuous. Compactness of $\mathbb{Z}_p$ tells us that f is uniformly continuous: for any given $\varepsilon = p^{-s}$ (s large), we have $t = t(s) > 0$ such that $|(k+p^t) - k|_p \le p^{-t(s)} \Rightarrow |f(k+p^t) - f(k)|_p < p^{-s}$. Since $\left|\binom{p^t}{k}\right|_p \le p^{-1}$ and $|\cdot|_p$ is non-Archimedean, we have

$$|a_{n+p^t}(f)|_p \le \max(p^{-1}|a_{n+1}(f)|_p, \ldots, p^{-1}|a_{n+p^t-1}(f)|_p, p^{-s}).$$

Since $|a_n(f)|_p \le 1$, we find if $n \ge p^{t(1)}$ $|a_n(f)|_p \le p^{-1}$. Then $|a_{n+p^{t(2)}}(f)| \le p^{-2}$ or equivalently, $|a_n(f)|_p \le p^{-2}$ if $n \ge p^{t(1)} + p^{t(2)}$, and repeating this process, inductively we get

$$|a_n(f)|_p \le p^{-m} \quad \text{if } n \ge p^{t(1)} + p^{t(2)} + \cdots + p^{t(m)}.$$

Thus we get the desired limit formula: $\lim_{n\to\infty} a_n(f) = 0$. $\square$

Exercise: Suppose $A = \mathbb{Q}_p$ and f is continuous. Use compacity of $\mathbb{Z}_p$, prove that there exists $0 < \alpha \in \mathbb{Z}$ such that $p^\alpha f$ has values in W, and using this fact, prove $\lim_{n\to\infty} a_n(f) = 0$.

§4.9. Space of continuous functions $f:\mathbb{Z}_p \to A$.

Let $C(\mathbb{Z}_p, A)$ be the space of all continuous functions $f:\mathbb{Z}_p \to A$. Then f has unique interpolation series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n(f)\binom{x}{n}$$

with $a_n(f) \in A$. Define the norm of f by $|f|_p := \sup_{x\in\mathbb{Z}_p} |f(x)|_p$. Plainly $|\cdot|_p$ is a norm satisfying $|f+g|_p \le \max(|f|_p, |g|_p)$, $|\alpha f|_p = |\alpha|_p |f|_p$ $(\alpha \in A)$ and $|f|_p = 0 \Leftrightarrow f = 0$.

Corollary 1. *We have* $|f|_p = \sup_n |a_n(f)|_p$

Proof. Note $|a_n(f)|_p = |\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(k)|_p \le \max_k |f(k)|_p \le |f|_p$, which shows $\sup_n |a_n(f)|_p \le |f|_p$. On the other hand, we have

$$|f|_p = |\lim_{N\to\infty}\sum_{n=0}^{N} a_n(f)\binom{x}{n}|_p = \lim_{N\to\infty}|\sum_{n=0}^{N} a_n(f)\binom{x}{n}|_p \le \sup_n |a_n(f)|_p$$

as desired. $\square$

§4.10. Space of p-adic measure on $\mathbb{Z}_p$.

Since uniform convergence preserves continuity, the space $C(\mathbb{Z}_p, A)$ is a Banach A-module (i.e., complete uner $|\cdot|_p$). We write $M(\mathbb{Z}_p, A)$ for the space of A-linear functional $\varphi : C(\mathbb{Z}_p, A) \to A$ such that $|\varphi(f)|_p \leq B|f|_p$ for a constant $B > 0$ independent of f. We often write $\int_{\mathbb{Z}_p} f d\varphi := \varphi(f)$. If $A = W$, then B can be taken to be 1. We define

$$|\varphi|_p = \sup_{0 \neq f \in C(\mathbb{Z}_p, A)} \frac{|\varphi(f)|_p}{|f|_p} \overset{(*)}{=} \sup_{f \in C(\mathbb{Z}_p, A), |f|_p = 1} |\varphi(f)|_p.$$

Exercise: Why the identity $(*)$ holds? Prove $|\varphi + \varphi'|_p \leq \max(|\varphi|_p, |\varphi'|_p)$, $|\alpha\varphi|_p = |\alpha|_p|\varphi|_p$ for $\alpha \in A$ and $|\varphi|_p = 0 \Leftrightarrow \varphi = 0$.

If $\lim_{n\to\infty} \varphi_n = \varphi$ under the above norm, we have $|\varphi|_p = \lim_{n\to\infty} |\varphi_n|$ and $|\varphi_m(f) - \varphi_n(f)|_p \leq |\varphi_m - \varphi_n|_p|f|_p$, and hence $\{\varphi_n(f)\}_n$ is a Cauchy sequence. Thus we define $\varphi(f) := \lim_{n\to\infty} \varphi_n(f)$. Therefore $|\varphi(f)|_p = \lim_{n\to\infty} |\varphi_n(f)|_p \leq \lim_{n\to\infty} |\varphi_n|_p|f|_p = |\varphi|_p|f|_p$. Thus $\varphi \in M(\mathbb{Z}_p, A)$, and $M(\mathbb{Z}_p, A)$ is a Banach A-module.

§4.11. Moment determines a measure.

Theorem 2. *For a given sequence* $\{b_n \in A\}_n$ *with bounded norm, there is a unique measure* $\varphi \in M(\mathbb{Z}_p, A)$ *satisfying* $\int \binom{x}{n} d\varphi = b_n$ *and* $|\varphi|_p = \sup_n |b_n|_p$ *such that if* $f = \sum_{n=0}^{\infty} a_n(f)\binom{x}{n} \in C(\mathbb{Z}_p, A)$*, we have* $\int f d\varphi = \sum_{n=0}^{\infty} b_n a_n(f)$*. All element* φ *in* $M(\mathbb{Z}_p, A)$ *is obtained this way.*

Proof. Since $\lim_{n\to\infty} a_n(f) = 0$ and $|b_n| \le B$ for all n, $\sum_{n=0}^{\infty} b_n a_n(f)$ converges, giving an A-linera map $\varphi : C(\mathbb{Z}_p, A) \to A$. Note

$$|\sum_{n=0}^{\infty} b_n a_n(f)| = \sup_n |b_n|_p |a_n(f)|_p \le B \sum_n |a_n(f)|_p = B|f|_p.$$

So $\varphi \in M(\mathbb{Z}_p, A)$. Taking $B := \sup_n |b_n|_p$, we find $|\varphi|_p \le B$. Since $|\int \binom{x}{n} d\varphi|_p = |b_n|_p$ and $|\binom{x}{n}|_p = 1$, we find $B = |\varphi|_p$. Since f has unique interpolation series expansion, every measure is given by the above way. $\square$

§**4.12. Corollary.** $\{\int x^n d\varphi\}_n$ **determines** $\varphi \in M(\mathbb{Z}_p, A)$**.**
By definition, we have $M(\mathbb{Z}_p, A) \subset M(\mathbb{Z}_p, K)$. Since $\binom{x}{n} = \sum_{j=0}^{n} a_j x^j$, the sequence $\{\int x^n d\varphi\}_n$ determines $\int \binom{x}{n} d\varphi = \sum_{j=0}^{n} a_j \int x^j d\varphi$ as an element of K. Then the above theorem implies φ is uniquely determined in $M(\mathbb{Z}_p, K)$. Since $M(\mathbb{Z}_p, A)$ is a subspace of $M(\mathbb{Z}_p, K)$, we find φ is determined by $\{\int x^n d\varphi\}_n \in A^{\mathbb{N}}$.

Consider a formal expansion $(1+T)^x = \sum_{n=0}^{\infty} \binom{x}{n} T^n$. For each $\varphi \in M(\mathbb{Z}_p, W)$, we define

$$\Phi_\varphi(T) := \int (1+T)^x d\varphi = \sum_{n=0}^{\infty} \int \binom{x}{n} d\varphi T^n \in W[[T]].$$

By the above facts, this gives an isomorphism

$$\Phi : M(\mathbb{Z}_p, W) \cong W[[T]].$$

§4.13. Ring structure of $M(\mathbb{Z}_p, W)$.

For two measures $\varphi, \psi \in M(\mathbb{Z}_p, A)$, we define $\int f d(\varphi * \psi) := \int\int f(x+y) d\varphi(x) d\psi(y)$. Then

$$\begin{aligned}\Phi_{\varphi*\psi} &= \int\int (1+T)^{x+y} d\varphi(x) d\psi(y) \\ &= \int (1+T)^x d\varphi(x) \cdot \int (1+T)^y d\psi(y) = \Phi_\varphi(T)\Phi_\psi(T).\end{aligned}$$

Thus $M(\mathbb{Z}_p, W)$ with convolution product $(\varphi, \psi) \mapsto \varphi * \psi$ is isomorphic to the power series ring $W[[T]]$. Defining $|\sum_{n=0}^\infty a_n T^n| = \sum_n |a_n|_p$, $W[[T]]$ is a Banach W-module. The theorem in §4.10 tells us

Corollary 2. *The isomorphism $\Phi : M(\mathbb{Z}_p, W) \cong W[[T]]$ is an isometry of normed rings.*

Exercise: What is the multiplicative identity of the ring $M(\mathbb{Z}_p, W)$?