We prove analytic continuation of Hecke L functions of number fields. A key is to write down explicitly the Hecke L-function as a linear combination of Shintani \( \zeta \)-function. For a number field, write \( F_+^\times \) for the group of all totally positive numbers in \( F \). Define, for linearly independent \( v_1, \ldots, v_r \in F_+^\times \), writing \( v = (v_1, \ldots, v_r) \), an open simplicial cone with generators \( v \) by

\[
C(v) = C(v_1, \ldots, v_r) := \mathbb{R}_+^\times v_1 + \cdots + \mathbb{R}_+^\times v_r
\]

\[
= \{ x_1 v_1 + \cdots + x_r v_r | x_i \in \mathbb{R}_+^\times \} \subset F \otimes_{\mathbb{Q}} \mathbb{R} =: F_\infty.
\]

For \( z, s \in \mathbb{C} \), writing \( z = |z|e^{i\theta} \) with \(-\pi < \theta \leq \pi\), we put \( z^s := |z|^s e^{i\theta s} \) in this notes. We start with the elementary case of quadratic fields and generalize the method to general number fields.
§3.0. Quadratic fields.
Let $F = \mathbb{Q}[\sqrt{d}]$ for square free integer $d$. It is known that the integer ring $O$ of $F$ is given by $O = \mathbb{Z}w_1 + \mathbb{Z}w_2$ with $w_1 = 1$ and

$$w_2 = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod{4}, \\ \frac{\sqrt{d}}{2} & \text{otherwise}. \end{cases}$$

Write $\sigma \in \text{Gal}(F/\mathbb{Q})$ given by $\sigma(a + b\sqrt{d}) = a - b\sqrt{d}$ (here $\sqrt{d}$ is normalized so that it is positive when $d > 0$ and it has positive imaginary part if $d < 0$). Let $F^\times_+$ be the group of totally positive elements in $F^\times$; so, if $F$ is imaginary, we understand $F^\times_+ = F^\times$.

Fix a non-zero $O$-ideal $n$. Let $O^\times_+(n) = \{\varepsilon \in O^\times_+ | \varepsilon \equiv 1 \pmod{n} \}$. If $F$ is real, choose $\varepsilon$ with $O^\times_+(n) = \varepsilon \mathbb{Z}$ with $\varepsilon < 1 < \varepsilon^\sigma$. If $F$ is imaginary, then $O^\times_+(n)$ is finite, and write $\varepsilon$ for a generator with non-negative imaginary part of the cyclic group $O^\times_+(n)$. 
§3.1. Cones in quadratic fields.

In the case where $\varepsilon \notin \{\pm 1\}$, we consider the cone $C(1, \varepsilon)$. Then every $\alpha \in F_+^\times$ is uniquely brought into $C := C(1, \varepsilon) \cup C(1)$ by the multiplication by an element $O_+^\times$. If $\varepsilon \in \{\pm 1\}$, we pick $\alpha_j \in F_+^\times (0 < j \leq k)$ with $\alpha_1 = 1$ so that $C = \bigsqcup_{j<k} C(\alpha_j) \cup \bigsqcup_{j<r} C(\alpha_j, \alpha_{j+1})$ has the same property. We may assume that if $C(v, w) \subset C$, the angle of $v$ and $w$ to be less than $90^\circ$. For an open cone $C(v, w)$, we have $\alpha C(v, w) = C(\alpha v, \alpha w)$. Multiplying generators by $\alpha$, we may assume that $C$ is a disjoint union of finitely many cones generated by $v$'s in $\mathfrak{a}$ with totally positive real part. For any $\xi \in C(v, w)$, there exists a unique $\alpha = av + bw \in C(v, w)$ with $(a, b) \in (0, 1]^2$ such that $\xi = (a+n_1)v+(b+n_2)w = \alpha + n_1 v + n_2 w$ for $(n_1, n_2) \in \mathbb{N}^2$. More generally, for a number field $F$ and $\xi \in C(v)$ with $v = t(v_1, \ldots, v_r) \in (F_+^\times)^r$, there exists a unique $\alpha = a_1 v_1 + \cdots + a_r v_r = n \cdot v \in C(v)$ with $(a_1, \ldots, a_r) \in (0, 1]^r$ such that

$$\xi = (a_1 + n_1) v_1 + \cdots + (a_r + n_r) v_r = \alpha + n \cdot v$$

for $n = (n_1, \ldots, n_r) \in \mathbb{N}^r$ (note $n \cdot v \in \mathfrak{a}$ if $v_j \in \mathfrak{a}$ for all $j$).
§3.2. Hecke L as a sum of partial \( \zeta \) function.

In this section, \( F \) is a general number field. Let \( a \) be an \( O \)-ideal prime to \( n \) and write \([a] \in Cl_F(n)\) for the ray class of \( \alpha \) modulo \( n \). Define a partial L-function of \([a]\) by

\[
\zeta_n(s, a) = \sum_{0 \neq b \subset O, [b] = [a^{-1}]} N(b)^{-s}.
\]

Formally, for a character \( \chi : Cl_F(n) \to \mathbb{C}^\times \), we have

\[
L(s, \chi) = \sum_{[a] \in Cl_F(n)} \chi(a)^{-1} \zeta_n(s, a).
\]

On the other hand, writing \( \alpha a^{-1} = b \) with \( \alpha \equiv 1 (\text{mod } n)^\times \), \( b \subset O \Leftrightarrow \alpha \in a \). Thus, indicating totally positivity of \( \alpha \) by \( \alpha \gg 0 \),

\[
\zeta_n(s, a) = N(a)^s \sum_{\alpha \in a/O_+^\times(n), \alpha \equiv 1 (\text{mod } n^\infty)^\times} \prod_{\sigma} (\alpha^\sigma)^{-s} = N(a)^s \sum_{0 \ll \alpha \in ((1+n) \cap a)/O_+^\times(n)} \prod_{\sigma} (\alpha^\sigma)^{-s},
\]

where \( \sigma \) runs over all field embeddings of \( F \) into \( \mathbb{C} \).
§3.3. Hecke $L$ as a sum of Shintani $\zeta$ for quadratic fields.

Pick $C(v)$ for $v = (v_1, \ldots, v_r) \subset \mathbb{C}$ with $r \leq 2$. Define

$$C(v) = \overline{C}_a(v) := \{a_1v_1 + \cdots + a_r v_r \in C(v) \cap a | a_k \in (0, 1]\}$$

which is a finite set as $a$ is a lattice in $F \otimes_{\mathbb{Q}} \mathbb{R}$.

Return to a quadratic field field $F$. Write $C = \bigsqcup_v C(v)$. As remarked already, we may assume $v \in na^r$ and $\Re(v_i) > 0$ and $\Re(v_i^\sigma) > 0$ for all $i$. Then for $A_v = (v, v^\sigma)$ and $\alpha = x \cdot v$ ($x \in (0, 1]^2$), $N(a)^{-s}\zeta_n(s, a)$ is given by

$$\sum_v \sum_{\alpha \in \overline{C}(v) \cap ((1+n)\cap a)} \sum_{n \in \mathbb{N}^r} (\alpha + n \cdot v)^{-s}(\alpha^\sigma + n \cdot v^\sigma)^{-s}$$

$$= \sum_v \sum_{\alpha \in \overline{C}(v) \cap ((1+n)\cap a)} \zeta((s, s), A_v, x\alpha, 1)$$

Thus $\zeta_n(s, a)$ and $L(s, \chi)$ have meromorphic continuation to all $s \in \mathbb{C}$. This is the original Shintani’s method in his 1976 paper.
3.4. Cone decomposition. Let $F$ be a general number field. An open cone $C(v)$ for $v = t(v_1, \ldots, v_n)$ in $F_+^\times$ spanned by $\mathbb{Q}$-linearly independent numbers $v_1, \ldots, v_n \in F_+^\times$ is defined to be

$$C(v) = \mathbb{R}_+^\times v_1 + \cdots + \mathbb{R}_+^\times v_n.$$ 

We will later prove

**Theorem 1** (D. Mumford 1975 and T. Shintani 1976). There exists finitely many open simplicial cones $C(v_1), \ldots, C(v_k)$ without intersection such that any $\alpha \in F_+^\times$, there exists a unique $\varepsilon \in O_+^\times(n)$ such that $\varepsilon \alpha \in C := \bigsqcup_{j=1}^k C(v_j)$ with $v_j \in \mathfrak{n}a$ and $\text{Re}(v_j^\sigma) > 0$ for all embeddings $\sigma : F \hookrightarrow \mathbb{C}$.

We admit this theorem for the moment. A disjoint union of open simplicial cones is called a *polyhedral cone*.

Mumford proved this to make smooth toroidal compactification of the Hilbert modular variety (and generalized this to $GL_n^+(O)$ with totally positive determinant in place of $O_+^\times = GL_1^+(O)$ to make a smooth toroidal compactification of Shimura varieties).
§3.5. Integral cone decomposition.

Write $C = \bigsqcup_v C(v)$ for $C$ as in the theorem. Pick an $O$-ideal $\mathfrak{a}$ prime to $\mathfrak{n}$. By multiplying by a positive integer, we may assume that $v_i \in \mathfrak{n}\mathfrak{a}$. Write $I = \text{Isom}_{\text{field}}(F, \mathbb{C})$ and $v = (v_1, \ldots, v_r)$ ($r \leq [F : \mathbb{Q}] = \dim_{\mathbb{Q}} F = m$). Ordering $I = \{\sigma_1, \ldots, \sigma_m\}$, define $A_v = (v^{\sigma_1}, \ldots, v^{\sigma_m}) \in M_{r,m}(\mathbb{C})$ and  

$$
\overline{C}(v) = \overline{C}_\mathfrak{a}(v) := \{a_1 v_1 + \cdots + a_r v_r \in C(v) \cap \mathfrak{a} | a_k \in (0, 1]\}.
$$

For any $\xi = a_1 v_1 + \cdots + a_r v_r \in C(v) \cap \mathfrak{a}$ prime to $\mathfrak{n}$, take the fraction part $\langle a_i \rangle \in [0, 1)$ such that $a_i - \langle a_i \rangle \in \mathbb{N}$. Then $\alpha = \alpha_\xi = \sum_{i=1}^r \langle a_i \rangle v_i \in \overline{C}(v)$, and we have, putting $x_\alpha := (\langle a_i \rangle)_i$, we have $\xi = (x_\alpha + n) \cdot v$ with $n \in \mathbb{N}^r$ (row vector). Thus $\xi \equiv \alpha \mod \mathfrak{n}$. and $\lambda((\xi))^{-k} = \lambda((\alpha_\xi))^{-k}$ for a Hecke character $\lambda$ modulo $\mathfrak{n}$ of weight $k$. As remarked in §3.1, the element $\alpha_\xi \in \overline{C}_\mathfrak{a}(v)$ is uniquely determined by $\xi \in \mathfrak{a}$. In other words, we have for $\mathfrak{a}_+ := \{\alpha \in \mathfrak{a} | \alpha \gg 0\}$,  

$$
\mathfrak{a}_+ / \mathcal{O}_+^X(\mathfrak{n}) \cong \bigsqcup_v \bigsqcup_{\alpha \in \overline{C}_\mathfrak{a}(v)} \{\alpha + n \cdot v | n \in \mathbb{N}^r\}
$$
§3.6. Meromorphic continuation for general $F$.

Let $\lambda$ be a Hecke character of weight $k$ modulo $n$. Define

$$\zeta(s, \lambda, a) = \sum_{0 \neq b \subset O, b \sim a^{-1}} \lambda(b)^{-1} N(b)^s = \frac{N(a)^s}{\lambda(a)} \sum_{(\xi) \subset a} \lambda(\alpha_\xi) \frac{\zeta^k}{\alpha_\xi^k} |N(\xi)|^{-s},$$

where $b \sim a^{-1}$ means the ideal class of $b$ in $Cl_F^+$ is equal to the class of $a^{-1}$. Since $\lambda(\xi)|N(\xi)|^{-s} = \zeta^{k-s1}$ if $\xi \gg 0$. We have

$$\frac{\lambda(a)}{N(a)^s} \zeta(s, \lambda, a) = \sum_{\xi \in a_+/O_+^\times} \frac{\lambda(\alpha_\xi)}{\alpha_\xi^k} \zeta^{k-s1} \zeta(s1 - k, A_v, x_\alpha, 1)$$

and

$$L(s, \lambda) = \sum_a \frac{N(a)^s}{\lambda(a)} \sum_v \sum_{\alpha \in C_a(v)} \frac{\lambda(\alpha)}{\alpha^k} \zeta(s1 - k, A_v, x_\alpha, 1),$$

which has meromorphic continuation.
§3.7. Known functional equation when $k = 0$. Write $\Sigma := I/\langle c \rangle$ for complex conjugation $c$, and decompose $\Sigma = \Sigma(\mathbb{R}) \sqcup \Sigma(\mathbb{C})$, where $\Sigma(\mathbb{R})$ is made of all real embeddings.

We say $\lambda$ is odd at $\sigma \in \Sigma(\mathbb{R})$ if $\lambda(\alpha) = -1$ for $\alpha \in F$ with $\alpha^\sigma < 0$ and $\alpha^\tau > 0$ for all $\tau \neq \sigma$ in $\Sigma(\mathbb{R})$ and $\alpha \equiv 1 \mod n$. Otherwise $\lambda$ is even at $\sigma$. Define $L_\sigma(s, \lambda) = \pi^{-s/2} \Gamma(\frac{s}{2})$ if $\lambda$ is even for $\sigma \in \Sigma(\mathbb{R})$, $L_\sigma(s, \lambda) = \pi^{-(s+1)/2} \Gamma(\frac{s+1}{2})$ if $\lambda$ is odd for $\sigma \in \Sigma(\mathbb{R})$ and $L_\sigma(s, \lambda) = (2\pi)^{-s} \Gamma(s)$ for $\sigma \in \Sigma(\mathbb{C})$. Define

$$\hat{L}(s, \lambda) = \left( \prod_{\sigma \in \Sigma} L_\sigma(s, \lambda) \right) L(s, \lambda).$$

Then for primitive $\lambda$, we have for $\kappa(\lambda) \in \bar{\mathbb{Q}}$ with $|\kappa(\lambda)| = 1$

$$\hat{L}(s, \lambda) = \kappa(\lambda)(|D|N(n))^{(1/2)-s} \hat{L}(1-s, \lambda^{-1}) \quad \text{(Hecke, 1917)}.$$ 

If $\lambda \neq 1$, $L(s, \lambda)$ is holomorphic everywhere, and $\hat{\zeta}_F(s)$ has simple pole at $s = 0, 1$. General $L(s, \lambda)$ has functional equation of the form $s \leftrightarrow w + 1 - s$ for $w \in \mathbb{Z}$ with $k + kc = w \sum_\sigma \sigma$ (see [LFE, §8.6]).
§3.8. Cone lemma for Mumford-Shintani theorem. Let $C$ and $C'$ be polyhedral cones in $F^\times_+$. Then $C \cap C'$, $C \cup C'$ and $C \setminus C'$ are all polyhedral cones.

**Sketch of Proof.** Since $C \cup C' = (C \setminus C') \sqcup (C \cap C') \sqcup (C' \setminus C)$, we need to prove this only for $C \cap C'$ and $C \setminus C'$. Since the complement $C^\perp$ of $C$ is a disjoint union of $C'(\pm v_1, \ldots, \pm v_{j-1}, \pm v_j, \pm v_{j+1}, \ldots, \pm v_r)$ with $^t(\pm v_1, \ldots, \pm v_{j-1}, \pm v_j, \pm v_{j+1}, \ldots, \pm v_r) \neq v$, we only need to prove the result for $C \cap C'$. We may assume that $C$ and $C'$ is simplicial; so, $C = C(v)$ and $C' = C(w)$. Then sending $C \ni a_1 v_1 + \cdots + a_r v_r$ to $(a_1, \ldots, a_r) \in (\mathbb{R}^\times_+)^r$, we have $C \cong (\mathbb{R}^\times_+)^r$. Thus we may assume that $C = (\mathbb{R}^\times_+)^r$ with $C \cap F^r$ sent to $(\mathbb{Q}^\times_+)^r$. Then decompose $C \cap C' = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k$ for connected components $C_j$. Then the closure $\overline{C}_j$ of $C_j$ has finitely many faces of the form $C(y_1, \ldots, y_{r-1})$. Take $y_r$ in $C_j$, then $\overline{C}(y_1, \ldots, y_{r-1}, y_r)$ covers $C_j$. Removing redundant faces, we get the decomposition. $\Box$
§3.9. Reduction towards Mumford-Shintani theorem.

Since the proof is the same for any \( n \), we treat \( O^\times_+ \). We may assume that \( O^\times_+ \) is infinite as otherwise, it is easy. Let \( V_+ := \prod_{\sigma \in \Sigma(\mathbb{R})} \mathbb{R}^\times_+ \times \prod_{\tau \in \Sigma(\mathbb{C})} \mathbb{C}^\times \). The space \( V_+ \) is a multiplicative abelian Lie group. Let

\[
X = \{ (v_\sigma)_{\sigma \in \Sigma} \in V_+ | N(x) = \prod_{\sigma \in \Sigma(\mathbb{R})} x_\sigma \times \prod_{\tau \in \Sigma(\mathbb{C})} |x_\sigma|^2 = 1 \}.
\]

Then \( X \) is a Lie subgroup of \( V_+ \). By \( \text{Log} : V_+ \to \mathbb{R}^{\Sigma} =: W \) given by \( \text{Log}(v_\sigma) = ((\log v_\sigma)_{\sigma \in \Sigma(\mathbb{R})}, (\log |v_\tau|^2)_{\tau \in \Sigma(\mathbb{C})}) \), \( \text{Log}(X) \) is a \( \mathbb{R} \)-vector subspace of \( W \), and by Dirichlet’s unit theorem, \( \text{Log}(O^\times_+) \) is a lattice of \( W \) (i.e., \( W/\text{Log}(X) \) is compact). We have an exact sequence, for \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \),

\[
1 \to (S^1)^{\Sigma(\mathbb{C})} \to X \to \text{Log}(X) \to 0.
\]

Thus we have a compact subset \( K \subset X \) such that

\[
X = \bigcup_{\varepsilon \in O^\times_+} \varepsilon K \text{ with the interior } K^\circ \text{ of } K \text{ containing 1.}
\]
§3.10. Proof of Mumford-Shintani theorem. We can then find an open subset $U \ni 1$ inside $K$ such that $\varepsilon U \cap U = \emptyset$ if $1 \neq \varepsilon O_+^\times$. Let $\pi : V_+ \to X$ be the projection $v \mapsto v/N(v)$. Since $F_+^\times$ is dense in $V_+$, $\pi(F_+^\times)$ is dense in $X$, which implies

$$K = \bigcup_{\alpha \in \pi(F_+^\times)} \alpha U.$$ 

Since $K$ is compact, there is a finite set $A \subset F_+^\times$ such that

$$K = \bigcup_{\alpha \in \pi(A)} \alpha U.$$ 

We may assume that $U = C_0 \cap X$ for an open simplicial cones $C_0$; so,

$$K = \bigcup_{\alpha \in A} \alpha C_0$$

as $\pi(\alpha)C = \alpha C_0$. Then by Cone Lemma, we can remove overlapping intersections and still $K$ is exactly the disjoint union of finitely many open simplicial cones. \qed
3.11. Towards integrality of L-values (Cassou-Nogues).
If $\Sigma(C) \neq \emptyset$, $L_\sigma(s, \lambda) = (2\pi)^{-s} \Gamma(s)$ for $\sigma \in \Sigma(C)$ has pole at $s = 1 - n$ ($0 < n \in \mathbb{Z}$), while $\hat{L}(n, \lambda^{-1}) < \infty$ and $L_\sigma(n, \lambda^{-1}) < \infty$; so, $L(1 - n, \lambda) = 0$ if $\Sigma(C) \neq \emptyset$. So we assume now that $F$ is totally real to study integrality of $L(1 - n, \lambda)$. Rationality is proven by Shintani in 1976, but here we describe Cassou-Nogues' method effective to show integrality also.

A generalized Bernoulli polynomial in \S 2.8 is given only for $\chi$ with $\chi_i \neq 1$ for all $i$. Thus we need to find a way to express $L(s, \lambda)$ as a linear combination of $\zeta(s, A_v, x_\alpha, \chi)$ for non-trivial $\chi$.

We take an integral ideal $a$ prime to $n$ and pick a prime ideal $l$ prime to $an$ such that $O/l \cong \mathbb{Z}/l\mathbb{Z}$ for a prime $l \in \mathbb{Z}$ (this means $(l)$ splits in $O$). There are such prime $l$ with positive density (actually the density $\geq 1/[F : \mathbb{Q}]$ by Chebotarev density theorem). We put $A = A_l := a/l\mathbb{A}$. Then $a/l \cong O/l \cong \mathbb{Z}/l\mathbb{Z}$ which is a cyclic group of prime order $l$. 
§3.12. Finite Fourier transform.

Let $\hat{A} := \text{Hom}(A, \mathbb{C}) = \text{Hom}(A, \mu_l)$. For functions $f : A \to \mathbb{C}$ and $g : \hat{A} \to \mathbb{C}$, define their Fourier transform to be

$$\mathcal{F}(f) = \hat{f}(\psi) = \sum_{a \in A} f(a) \psi(a) \quad \text{and} \quad \mathcal{F}(g)(x) = \sum_{\psi \in \hat{A}} g(\psi) \psi(x).$$

Exercise: If $F = \mathbb{Q}$ and $a = \mathbb{Z}$, for a Dirichlet character $\chi$ modulo $l$, show $\hat{\chi}(\psi) = G(\chi)$ for $\psi(x) = e(\frac{2\pi i x}{l})$, and find an explicit formula of $\mathcal{F}(\mathcal{F}(f)) : A \to \mathbb{C}$.

Lemma 1. For $1 : \hat{A} \to \mathbb{C}$ with $1(\psi) = 1$ for $\psi \neq 1$ and $1(1) = 0$.

Then $\mathcal{F}(1)(x) = -\begin{cases} 1 & \text{if } x \neq 0, \\ 1 - l & \text{if } x = 0. \end{cases}$

Proof. If $x = 0$, the $\mathcal{F}(1) = |\hat{A} - \{1\}| = l - 1$. If $x \neq 0$, $\mathcal{F}(1)$ is the sum of all $l$-th roots of unity except for 1. Since $\sum_{\zeta \in \mu_l} \zeta = 0$, we get $\mathcal{F}(1)(x) = -1$. \(\square\)
§3.13. A Shintani zeta function with $\psi$.

Let $\chi_{v,\psi} = (\psi(v_i))$. Then $\chi_{v,\psi}^n = \prod_i \psi(v_i)^{n_i} = \psi(n_1v_1 + \cdots + n_rv_r)$.

Recall

$$
\zeta(s, A_v, x_\alpha, \chi_v, \psi) = \sum_{n \in \mathbb{Z}_+^r} \chi_{v,\psi}^n \prod_j (\alpha^\sigma_j + n_1v_1^\sigma_j + \cdots + n_rv_r^\sigma_j)^{-s_j}.
$$

By the lemma, we have

$$
\sum_{\psi \neq 1} \prod_{\alpha \in \mathcal{C}_a(v)} \lambda(\alpha) \zeta(s, A_v, x_\alpha, \chi_v, \psi)
= - \sum_{\alpha \in \mathcal{C}_a(v)} \lambda(\alpha) \zeta(s, A_v, x_\alpha, 1)
+ l \sum_{\beta \in \mathcal{C}_{a_l}(v)} \lambda(\beta) \zeta(s, A_v, x_\beta, 1)
$$

By choosing $l$ sufficiently large, we may assume $v_i \notin la$; i.e., $\psi(v_i) \neq 1$ for all $\psi \neq 1$. 
§3.14. Conclusion.

Multiplying by $N(a)^s \lambda(a)^{-1}$ and summing over $v$ and $a$, we get

$$\sum_a \frac{N(a)^s}{\lambda(a)} \sum_v \sum_{\psi \neq 1} \lambda(\alpha) \zeta(s, A_v, x_\alpha, \chi_\psi)$$

$$= - \sum_{a,v} \frac{N(a)^s}{\lambda(a)} \sum_{\alpha \in C_a(v)} \lambda(\alpha) \zeta(s, A_v, x_\alpha, 1)$$

$$+ l \sum_{a,v} \frac{N(a)^s}{\lambda(a)} \sum_{\beta \in C_a(lv)} \lambda(\beta) \zeta(s, A_v, x_\beta, 1)$$

$$= - \sum_{a,v} \frac{N(a)^s}{\lambda(a)} \sum_{\alpha \in C_a(v)} \lambda(\alpha) \zeta(s, A_v, x_\alpha, 1)$$

$$+ \frac{\lambda(l)}{N(l)} \sum_{a,v} \frac{N(la)^s}{\lambda(la)} \sum_{\beta \in C_a(lv)} \lambda(\beta) \zeta(s, A_v, x_\beta, 1)$$

$$\overset{(*)}{=} -(1 - \lambda(l)N(l)^{1-s})L(s, \lambda).$$

The identity at $(*)$ is because $\{la\}_a \cong Cl_{F}^{+}$. 
§3.15. If $\chi$ is non-trivial, no variable change necessary. Recall that $F(z, A, x, \chi) = \prod_{i=1}^{r} \frac{\exp(-x_i L_i(z))}{1 - \chi_i \exp(-L_i(z))}$ has pole at $L_i(z) = \log \chi + 2\pi i \mathbb{Z}$. If $\chi_i = |\chi_i| e^{i\theta_i}$ with $0 \neq \theta_i \in (-\pi, \pi)$ or $|\chi| < 1$, the pole avoid $z = 0$. Thus the poles avoid original contour $P(\varepsilon)^r$ in $z$-space. Therefore by the above trick, even to make analytic continuation of $L(s, \lambda)$, Shintani’s variable change is not necessary (as long as $\chi_i \neq 1$ for all $i$).

The corresponding rational function are therefore, writing $t_i = \exp(-L_i(z))$, of the form

$$\prod_{i=1}^{r} \frac{tx_i}{(1 - \psi(v_i) t_i)},$$

where $x_\alpha = (x_1, \ldots, x_r) \in [0, 1]^r \cap \mathbb{Q}^r$. 

§3.16. Rationality and integrality theorem. For an embedding $\sigma : F \hookrightarrow \mathbb{R}$, let $\alpha_\sigma \in F^\times$ be an element such that $\alpha_\sigma \equiv 1 \pmod{\mathfrak{n}}$ and $\alpha_\sigma^\tau < 0$ but $\alpha_\sigma^\tau > 0$ for all embedding $\tau$ other than $\sigma$. The character $\lambda$ modulo $\mathfrak{n}$ is called totally odd (resp. totally even) if $\lambda(\alpha_\sigma) = -1$ (resp. $\lambda(\alpha_\sigma) = 1$) for all field embeddings $\sigma$ of $F$.

Theorem 2 (Siegel 1937, Klingen 1962, Shintani 1976, Pierrete Cassou-Nogues 1979). Let $F \neq \mathbb{Q}$ be a totally real number field and $\lambda$ be a finite order ray class character. For $0 < n \in \mathbb{Z}$ and a split prime $l$ of $F$ outside $\mathfrak{n}$ with sufficiently large $N(l)$,

$$(1 - \lambda(l)N(l)^n)L(1 - n, \lambda) \in \mathbb{Z}[\lambda],$$

where $\mathbb{Z}[\lambda]$ is the subring of $\mathbb{C}$ generated by the values of $\lambda$. We have $L(1 - n, \lambda) \neq 0$ only when (i) $\lambda$ is totally odd and $n$ is odd or (ii) $\lambda$ is totally even and $n$ is even.

Exercise: Use functional equation to show vanishing of $L(1 - n, \lambda)$ when the condition of the theorem is not met.