Lecture note No.3 for Math 205a Fall 2019

Hecke L-function.

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We prove analytic continuation of Hecke L functions of number fields. A key is to write down explicitly the Hecke L-function as a linear combination of Shintani ζ-function. For a number field, write $F_+^\times$ for the group of all totally positive numbers in F. Define, for linearly independent $v_1, \dots, v_r \in F_+^\times$, writing $\mathbf{v} = (v_1, \dots, v_r)$, an *open simplicial cone* with generators $\mathbf{v}$ by

$$\begin{aligned} C(\mathbf{v}) = C(v_1, \dots, v_r) &:= \mathbb{R}_+^\times v_1 + \cdots + \mathbb{R}_+^\times v_r \\ &= \{x_1 v_1 + \cdots + x_r v_r | x_i \in \mathbb{R}_+^\times\} \subset F \otimes_{\mathbb{Q}} \mathbb{R} =: F_\infty. \end{aligned}$$

For $z, s \in \mathbb{C}$, writing $z = |z|e^{i\theta}$ with $-\pi < \theta \le \pi$, we put $z^s := |z|^s e^{i\theta s}$ in this notes. We start with the elementary case of quadratic fields and generalize the method to general number fields.

§3.0. Quadratic fields.

Let $F = \mathbb{Q}[\sqrt{d}]$ for square free integer d. It is known that the integer ring O of F is given by $O = \mathbb{Z}w_1 + \mathbb{Z}w_2$ with $w_1 = 1$ and

$$w_2 = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4, \\ \sqrt{d} & \text{otherwise.} \end{cases}$$

Write $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ given by $\sigma(a + b\sqrt{d}) = a - b\sqrt{d}$ (here $\sqrt{d}$ is normalized so that it is positive when $d > 0$ and it has positive imaginary part if $d < 0$. Let $F_+^\times$ be the group of totally positive elements in $F^\times$; so, if F is imaginary, we understand $F_+^\times = F^\times$.

Fix a non-zero O-ideal $\mathfrak{n}$. Let $O_+^\times(\mathfrak{n}) = \{\varepsilon \in O_+^\times | \varepsilon \equiv 1 \mod \mathfrak{n}\}$. If F is real, choose ε with $O_+^\times(\mathfrak{n}) = \varepsilon^{\mathbb{Z}}$ with $\varepsilon < 1 < \varepsilon^\sigma$. If F is imaginary, then $O_+^\times(\mathfrak{n})$ is finite, and write ε for a generator with non-negative imaginary part of the cyclic group $O_+^\times(\mathfrak{n})$.

§3.1. Cones in quadratic fields.

In the case where $\varepsilon \notin \{\pm 1\}$, we consider the cone $C(1,\varepsilon)$. Then every $\alpha \in F_+^\times$ is uniquely brought into $C := C(1,\varepsilon) \cup C(1)$ by the multiplication by an element $O_+^\times$. If $\varepsilon \in \{\pm 1\}$, we pick $\alpha_j \in F_+^\times$ $(0 < j \leq k)$ with $\alpha_1 = 1$ so that $C = \bigsqcup_{j<k} C(\alpha_j) \sqcup \bigsqcup_{j<r} C(\alpha_j, \alpha_{j+1})$ has the same property. We may assume that if $C(v,w) \subset C$, the angle of v and w to be less than 90°. For an open cone $C(v,w)$, we have $\alpha C(v,w) = C(\alpha v, \alpha w)$. Multiplying generators by α, we may assume that C is a disjoint union of finitely many cones generated by v's in $\mathfrak{n}\mathfrak{a}$ with totally positive real part. For any $\xi \in C(v,w)$, there exists a unique $\alpha = av + bw \in C(v,w)$ with $(a,b) \in (0,1]^2$ such that $\xi = (a+n_1)v + (b+n_2)w = \alpha + n_1 v + n_2 w$ for $(n_1, n_2) \in \mathbb{N}^2$. More generally, for a number field F and $\xi \in C(\mathbf{v})$ with $\mathbf{v} = {}^t(v_1, \ldots, v_r) \in (F_+^\times)^r$, there exists a unique $\alpha = a_1 v_1 + \cdots + a_r v_r = n \cdot \mathbf{v} \in C(\mathbf{v})$ with $(a_1, \ldots, a_r) \in (0,1]^r$ such that

$$\xi = (a_1 + n_1)v_1 + \cdots + (a_r + n_r)v_r = \alpha + n \cdot \mathbf{v}$$

for $n = (n_1, \ldots, n_r) \in \mathbb{N}^r$ (note $n \cdot \mathbf{v} \in \mathfrak{n}\mathfrak{a}$ if $v_j \in \mathfrak{n}\mathfrak{a}$ for all j).

§3.2. Hecke L as a sum of partial ζ function.

In this section, F is a general number field. Let $\mathfrak{a}$ be an O-ideal prime to $\mathfrak{n}$ and write $[\mathfrak{a}] \in Cl_F(\mathfrak{n})$ for the ray class of α modulo $\mathfrak{n}$. Define a partial L-function of $[\mathfrak{a}]$ by

$$\zeta_{\mathfrak{n}}(s, \mathfrak{a}) = \sum_{0 \neq \mathfrak{b} \subset O, [\mathfrak{b}] = [\mathfrak{a}^{-1}]} N(\mathfrak{b})^{-s}.$$

Formally, for a character $\chi : Cl_F(\mathfrak{n}) \to \mathbb{C}^\times$, we have

$$L(s, \chi) = \sum_{[\mathfrak{a}] \in Cl_F(\mathfrak{n})} \chi(\mathfrak{a})^{-1} \zeta_{\mathfrak{n}}(s, \mathfrak{a}).$$

On the other hand, writing $\alpha\mathfrak{a}^{-1} = \mathfrak{b}$ with $\alpha \equiv 1 (\text{mod } \mathfrak{n})^\times$, $\mathfrak{b} \subset O \Leftrightarrow \alpha \in \mathfrak{a}$. Thus, indicating totally positivity of α by $\alpha \gg 0$,

$$\begin{aligned}\zeta_{\mathfrak{n}}(s, \mathfrak{a}) = N(\mathfrak{a})^s & \sum_{\alpha \in \mathfrak{a}/O_+^\times(\mathfrak{n}), \alpha \equiv 1 (\text{mod } \mathfrak{n}\infty)^\times} \prod_\sigma (\alpha^\sigma)^{-s} \\ &= N(\mathfrak{a})^s \sum_{0 \ll \alpha \in ((1+\mathfrak{n}) \cap \mathfrak{a})/O_+^\times(\mathfrak{n})} \prod_\sigma (\alpha^\sigma)^{-s},\end{aligned}$$

where σ runs over all field embeddings of F into $\mathbb{C}$.

§3.3. Hecke L as a sum of Shintani ζ for quadratic fields. Pick $C(\mathbf{v})$ for $\mathbf{v} = {}^t(v_1, \ldots, v_r) \subset C$ with $r \leq 2$. Define

$$\overline{C}(\mathbf{v}) = \overline{C}_{\mathfrak{a}}(\mathbf{v}) := \{a_1 v_1 + \cdots + a_r v_r \in C(\mathbf{v}) \cap \mathfrak{a} | a_k \in (0,1]\}$$

which is a finite set as $\mathfrak{a}$ is a lattice in $F \otimes_{\mathbb{Q}} \mathbb{R}$.

Return to a quadratic field field F. Write $C = \bigsqcup_{\mathbf{v}} C(\mathbf{v})$. As remarked already, we may assume $\mathbf{v} \in \mathfrak{n}\mathfrak{a}^r$ and Re$(v_i) > 0$ and Re$(v_i^\sigma) > 0$ for all i. Then for $A_{\mathbf{v}} = (\mathbf{v}, \mathbf{v}^\sigma)$ and $\alpha = x \cdot v$ ($x \in (0,1]^2$), $N(\mathfrak{a})^{-s}\zeta_{\mathfrak{n}}(s, \mathfrak{a})$ is given by

$$\sum_{\mathbf{v}} \sum_{\alpha \in \overline{C}(\mathbf{v}) \cap ((1+\mathfrak{n}) \cap \mathfrak{a})} \sum_{n \in \mathbb{N}^r} (\alpha + n \cdot \mathbf{v})^{-s} (\alpha^\sigma + n \cdot \mathbf{v}^\sigma)^{-s}$$

$$= \sum_{\mathbf{v}} \sum_{\alpha \in \overline{C}(\mathbf{v}) \cap ((1+\mathfrak{n}) \cap \mathfrak{a})} \zeta((s,s), A_{\mathbf{v}}, x_\alpha, \mathbf{1})$$

Thus $\zeta_{\mathfrak{n}}(s, \mathfrak{a})$ and $L(s, \chi)$ have meromorphic continuation to all $s \in \mathbb{C}$. This is the original Shintani's method in his 1976 paper.

§**3.4. Cone decomposition.** Let F be a general number field. An open cone $C(\mathbf{v})$ for $\mathbf{v} = {}^t(v_1, \dots, v_n)$ in $F_+^\times$ spanned by $\mathbb{Q}$-linearly independent numbers $v_1, \dots, v_n \in F_+^\times$ is defined to be

$$C(\mathbf{v}) = \mathbb{R}_+^\times v_1 + \cdots + \mathbb{R}_+^\times v_n.$$

We will later prove

Theorem 1 (D. Mumford 1975 and T. Shintani 1976). *There exists finitely many open simplicial cones $C(\mathbf{v}_1), \dots, C(\mathbf{v}_k)$ without intersection such that any $\alpha \in F_+^\times$, there exists a unique $\varepsilon \in O_+^\times(\mathfrak{n})$ such that $\varepsilon\alpha \in C := \bigsqcup_{j=1}^k C(\mathbf{v}_j)$ with $v_j \in \mathfrak{n}\mathfrak{a}$ and* Re$(v_j^\sigma) > 0$ *for all embeddings $\sigma : F \hookrightarrow \mathbb{C}$.*

We admit this theorem for the moment. A disjoint union of open simplicial cones is called a *polyhedral cone*.

Mumford proved this to make smooth toroidal compactification of the Hilbert modular variety (and generalized this to $GL_n^+(O)$ with totally positive determinant in place of $O_+^\times = \mathrm{GL}_1^+(O)$ to make a smooth toroidal compactification of Shimura varieties).

§3.5. Integral cone decomposition.
Write $C = \bigsqcup_{\mathbf{v}} C(\mathbf{v})$ for C as in the theorem. Pick an O-ideal $\mathfrak{a}$ prime to $\mathfrak{n}$. By multiplying by a positive integer, we may assume that $v_i \in \mathfrak{n}\mathfrak{a}$. Write $I = \mathrm{Isom}_{\mathsf{field}}(F, \mathbb{C})$ and $\mathbf{v} = {}^t(v_1, \dots, v_r)$ $(r \le [F : \mathbb{Q}] = \dim_{\mathbb{Q}} F = m)$. Ordering $I = \{\sigma_1, \dots, \sigma_m\}$, define $A_{\mathbf{v}} = (\mathbf{v}^{\sigma_1}, \dots, \mathbf{v}^{\sigma_m}) \in M_{r,m}(\mathbb{C})$ and

$$\overline{C}(\mathbf{v}) = \overline{C}_{\mathfrak{a}}(\mathbf{v}) := \{a_1 v_1 + \cdots + a_r v_r \in C(\mathbf{v}) \cap \mathfrak{a} | a_k \in (0, 1]\}.$$

For any $\xi = a_1 v_1 + \cdots + a_r v_r \in C(\mathbf{v}) \cap \mathfrak{a}$ prime to $\mathfrak{n}$, take the fraction part $\langle a_i \rangle \in [0, 1)$ such that $a_i - \langle a_i \rangle \in \mathbb{N}$. Then $\alpha = \alpha_\xi = \sum_{i=1}^r \langle a_i \rangle v_i \in \overline{C}(\mathbf{v})$, and we have, putting $x_\alpha := (\langle a_i \rangle)_i$, we have $\xi = (x_\alpha + n) \cdot \mathbf{v}$ with $n \in \mathbb{N}^r$ (row vector). Thus $\xi \equiv \alpha \mod \mathfrak{n}$, and $\lambda((\xi))\xi^{-k} = \lambda((\alpha_\xi))\alpha_\xi^{-k}$ for a Hecke character λ modulo $\mathfrak{n}$ of weight k. As remarked in §3.1, the element $\alpha_\xi \in \overline{C}_{\mathfrak{a}}(\mathbf{v})$ is uniquely determined by $\xi \in \mathfrak{a}$. In other words, we have for $\mathfrak{a}_+ := \{\alpha \in \mathfrak{a} | \alpha \gg 0\}$,

$$\mathfrak{a}_+ / O_+^\times(\mathfrak{n}) \cong \bigsqcup_{\mathbf{v}} \bigsqcup_{\alpha \in \overline{C}_{\mathfrak{a}}(\mathbf{v})} \{\alpha + n \cdot {}^t\mathbf{v} | n \in \mathbb{N}^r\}$$

§3.6. Meromorphic continuation for general F.

Let λ be a Hecke character of weight k modulo $\mathfrak{n}$. Define

$$\zeta(s,\lambda,\mathfrak{a}) = \sum_{0\neq\mathfrak{b}\subset O,\mathfrak{b}\sim\mathfrak{a}^{-1}} \lambda(\mathfrak{b})^{-1}N(\mathfrak{b})^s = \frac{N(\mathfrak{a})^s}{\lambda(\mathfrak{a})}\sum_{(\xi)\subset\mathfrak{a}} \lambda(\alpha_\xi)\frac{\xi^k}{\alpha_\xi^k}|N(\xi)|^{-s},$$

where $\mathfrak{b}\sim\mathfrak{a}^{-1}$ means the ideal class of $\mathfrak{b}$ in Cl_F^+ is equal to the class of $\mathfrak{a}^{-1}$. Since $\lambda(\xi)|N(\xi)|^{-s} = \xi^{k-s\mathbf{1}}$ if $\xi \gg 0$. We have

$$\begin{aligned}\frac{\lambda(\mathfrak{a})}{N(\mathfrak{a})^s}\zeta(s,\lambda,\mathfrak{a}) &= \sum_{\xi\in\mathfrak{a}_+/O_+^\times} \frac{\lambda(\alpha_\xi)}{\alpha_\xi^k}\xi^{k-s\mathbf{1}} \\ &= \sum_{\mathbf{v}}\sum_{\alpha\in\overline{C}_{\mathfrak{a}}(\mathbf{v})} \frac{\lambda(\alpha)}{\alpha^k}\zeta(s\mathbf{1}-k,A_{\mathbf{v}},x_\alpha,\mathbf{1})\end{aligned}$$

and

$$L(s,\lambda) = \sum_{\mathfrak{a}}\frac{N(\mathfrak{a})^s}{\lambda(\mathfrak{a})}\sum_{\mathbf{v}}\sum_{\alpha\in\overline{C}_{\mathfrak{a}}(\mathbf{v})} \frac{\lambda(\alpha)}{\alpha^k}\zeta(s\mathbf{1}-k,A_{\mathbf{v}},x_\alpha,\mathbf{1}),$$

which has meromorphic continuation.

§**3.7. Known functional equation when** $k = 0$**.** Write $\Sigma := I/\langle c\rangle$ for complex conjugation c, and decompose $\Sigma = \Sigma(\mathbb{R}) \sqcup \Sigma(\mathbb{C})$, where $\Sigma(\mathbb{R})$ is made of all real embeddings.

We say λ is odd at $\sigma \in \Sigma(\mathbb{R})$ if $\lambda(\alpha) = -1$ for $\alpha \in F$ with $\alpha^\sigma < 0$ and $\alpha^\tau > 0$ for all $\tau \neq \sigma$ in $\Sigma(\mathbb{R})$ and $\alpha \equiv 1 \mod \mathfrak{n}$. Otherwise λ is even at σ. Define $L_\sigma(s, \lambda) = \pi^{-s/2}\Gamma(\frac{s}{2})$ if λ is even for $\sigma \in \Sigma(\mathbb{R})$, $L_\sigma(s, \lambda) = \pi^{-(s+1)/2}\Gamma(\frac{s+1}{2})$ if λ is odd for $\sigma \in \Sigma(\mathbb{R})$ and $L_\sigma(s, \lambda) = (2\pi)^{-s}\Gamma(s)$ for $\sigma \in \Sigma(\mathbb{C})$. Define

$$\widehat{L}(s, \lambda) = (\prod_{\sigma\in\Sigma} L_\sigma(s, \lambda))L(s, \lambda).$$

Then for primitive λ, we have for $\kappa(\lambda) \in \overline{\mathbb{Q}}$ with $|\kappa(\lambda)| = 1$

$$\widehat{L}(s, \lambda) = \kappa(\lambda)(|D|N(\mathfrak{n}))^{(1/2)-s}\widehat{L}(1 - s, \lambda^{-1}) \quad \text{(Hecke, 1917)}.$$

If $\lambda \neq \mathbf{1}$, $L(s, \lambda)$ is holomorphic everywhere, and $\widehat{\zeta}_F(s)$ has simple pole at $s = 0, 1$. General $L(s, \lambda)$ has functional equation of the form $s \leftrightarrow w + 1 - s$ for $w \in \mathbb{Z}$ with $k + kc = w\sum_\sigma \sigma$ (see [LFE, §8.6]).

§3.8. Cone lemma for Mumford-Shintani theorem. *Let C and C' be polyhedral cones in $F_+^\times$. Then $C \cap C'$, $C \cup C'$ and $C \setminus C'$ are all polyhedral cones.*

Sketch of Proof. Since $C \cup C' = (C \setminus C') \sqcup (C \cap C') \sqcup (C' \setminus C)$, we need to prove this only for $C \cap C'$ and $C \setminus C'$. Since the complement $C^\perp$ of C is a disjoint union of $C(\pm v_1, \dots, \pm v_{j-1}, \pm v_j, \pm v_{j+1}, \dots, \pm v_r)$ with ${}^t(\pm v_1, \dots, \pm v_{j-1}, \pm v_j, \pm v_{j+1}, \dots, \pm v_r) \neq \mathbf{v}$, we only need to prove the result for $C \cap C'$. We may assume that C and C' is simplicial; so, $C = C(\mathbf{v})$ and $C' = C(\mathbf{w})$. Then sending $C \ni a_1 v_1 + \cdots + a_r v_r$ to $(a_1, \dots, a_r) \in (\mathbb{R}_+^\times)^r$, we have $C \cong (\mathbb{R}_+^\times)^r$. Thus we may assume that $C = (\mathbb{R}_+^\times)^r$ with $C \cap F^r$ sent to $(\mathbb{Q}_+^\times)^r$. Then decompose $C \cap C' = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k$ for connected components C_j. Then the closure $\overline{C}_j$ of C_j has finitely many faces of the form $C(y_1, \dots, y_{r-1})$. Take y_r in C_j, then $\overline{C}(y_1, \dots, y_{r-1}, y_r)$ covers C_j. Removing redundant faces, we get the decomposition. $\square$

§3.9. Reduction towards Mumford-Shintani theorem.

Since the proof is the same for any $\mathfrak{n}$, we treat $O_+^\times$. We may assume that $O_+^\times$ is infinite as otherwise, it is easy. Let $V_+ := \prod_{\sigma\in\Sigma(\mathbb{R})}\mathbb{R}_+^\times \times \prod_{\tau\in\Sigma(\mathbb{C})}\mathbb{C}^\times$. The space V_+ is a multiplicative abelian Lie group. Let

$$X = \{(v_\sigma)_{\sigma\in\Sigma} \in V_+ | N(x) = \prod_{\sigma\in\Sigma(\mathbb{R})} x_\sigma \times \prod_{\tau\in\Sigma(\mathbb{C})} |x_\sigma|^2 = 1\}.$$

Then X is a Lie subgroup of V_+. By $\mathrm{Log} : V_+ \to \mathbb{R}^\Sigma =: W$ given by $\mathrm{Log}(v_\sigma) = ((\log v_\sigma)_{\sigma\in\Sigma(\mathbb{R})}, (\log |v_\tau|^2)_{\tau\in\Sigma(\mathbb{C})})$, $\mathrm{Log}(X)$ is a $\mathbb{R}$-vector subspace of W, and by Dirichlet's unit theorem, $\mathrm{Log}(O_+^\times)$ is a lattice of W (i.e., $W/\mathrm{Log}(X)$ is compact). We have an exact sequence, for $S^1 = \{z \in \mathbb{C} : |z| = 1\}$,

$$1 \to (S^1)^{\Sigma(\mathbb{C})} \to X \to \mathrm{Log}(X) \to 0.$$

Thus we have a compact subset $K \subset X$ such that

$$X = \bigcup_{\varepsilon\in O_+^\times} \varepsilon K \quad \text{with the interior } K^\circ \text{ of } K \text{ containing } 1.$$

§3.10. Proof of Mumford-Shintani theorem. We can then find an open subset $U \ni 1$ inside K such that $\varepsilon U \cap U = \emptyset$ if $1 \neq \varepsilon O_+^\times$. Let $\pi : V_+ \twoheadrightarrow X$ be the projection $v \mapsto v/N(v)$. Since $F_+^\times$ is dense in V_+, $\pi(F_+^\times)$ is dense in X, which implies

$$K = \bigcup_{\alpha \in \pi(F_+^\times)} \alpha U.$$

Since K is compact, there is a finite set $A \subset F_+^\times$ such that

$$K = \bigcup_{\alpha \in \pi(A)} \alpha U.$$

We may assume that $U = C_0 \cap X$ for an open simplicial cones C_0; so,

$$K = \bigcup_{\alpha \in A} \alpha C_0$$

as $\pi(\alpha)C = \alpha C_0$. Then by Cone Lemma, we can remove overlapping intersections and still K is exactly the disjoint union of finitely many open simplicial cones. $\square$

§**3.11. Towards integrality of L-values** (Cassou-Nogues).
If $\Sigma(\mathbb{C}) \neq \emptyset$, $L_\sigma(s,\lambda) = (2\pi)^{-s}\Gamma(s)$ for $\sigma \in \Sigma(\mathbb{C})$ has pole at $s = 1-n$ $(0 < n \in \mathbb{Z})$, while $\widehat{L}(n,\lambda^{-1}) < \infty$ and $L_\sigma(n,\lambda^{-1}) < \infty$; so, $L(1-n,\lambda) = 0$ if $\Sigma(\mathbb{C}) \neq \emptyset$. So we assume now that F is totally real to study integrality of $L(1-n,\lambda)$. Rationality is proven by Shintani in 1976, but here we describe Cassou-Nogues' method effective to show integrality also.

A generalized Bernoulli polynomial in §2.8 is given only for χ with $\chi_i \neq 1$ for all i. Thus we need to find a way to express $L(s,\lambda)$ as a linear combination of $\zeta(s, A_{\mathbf{v}}, x_\alpha, \chi)$ for non-trivial χ.

We take an integral ideal $\mathfrak{a}$ prime to $\mathfrak{n}$ and pick a prime ideal $\mathfrak{l}$ prime to $\mathfrak{a}\mathfrak{n}$ such that $O/\mathfrak{l} \cong \mathbb{Z}/l\mathbb{Z}$ for a prime $l \in \mathbb{Z}$ (this means (l) splits in O). There are such prime l with positive density (actually the density $\geq 1/[F:\mathbb{Q}]$ by Chebotarev density theorem). We put $\mathcal{A} = \mathcal{A}_\mathfrak{l} := \mathfrak{a}/\mathfrak{l}\mathfrak{a}$. Then $\mathfrak{a}/\mathfrak{l} \cong O/\mathfrak{l} \cong \mathbb{Z}/l\mathbb{Z}$ which is a cyclic group of prime order l.

§3.12. Finite Fourier transform.

Let $\widehat{\mathcal{A}} := \mathrm{Hom}(\mathcal{A}, \mathbb{C}) = \mathrm{Hom}(\mathcal{A}, \mu_l)$. For functions $f : \mathcal{A} \to \mathbb{C}$ and $g : \widehat{\mathcal{A}} \to \mathbb{C}$, define their Fourier transform to be

$$\mathcal{F}(f) = \widehat{f}(\psi) = \sum_{a \in \mathcal{A}} f(a)\psi(a) \text{ and } \mathcal{F}(g)(x) = \sum_{\psi \in \widehat{\mathcal{A}}} g(\psi)\psi(x).$$

Exercise: If $F = \mathbb{Q}$ and $\mathfrak{a} = \mathbb{Z}$, for a Dirichlet character χ modulo l, show $\widehat{\chi}(\psi) = G(\chi)$ for $\psi(x) = \mathbf{e}(\frac{2\pi i x}{l})$, and find an explicit formula of $\mathcal{F}(\mathcal{F}(f)) : \mathcal{A} \to \mathbb{C}$.

Lemma 1. *For* $\mathbf{1} : \widehat{\mathcal{A}} \to \mathbb{C}$ *with* $\mathbf{1}(\psi) = 1$ *for* $\psi \neq 1$ *and* $\mathbf{1}(1) = 0$. *Then* $\mathcal{F}(\mathbf{1})(x) = -\begin{cases} 1 & \text{if } x \neq 0, \\ 1 - l & \text{if } x = 0. \end{cases}$

Proof. If $x = 0$, the $\mathcal{F}(\mathbf{1}) = |\widehat{\mathcal{A}} - \{1\}| = l - 1$. If $x \neq 0$, $\mathcal{F}(\mathbf{1})$ is the sum of all l-th roots of unity except for 1. Since $\sum_{\zeta \in \mu_l} \zeta = 0$, we get $\mathcal{F}(\mathbf{1})(x) = -1$. □

§3.13. A Shintani zeta function with ψ.

Let $\chi_{\mathbf{v},\psi} = (\psi(v_i))$. Then $\chi_{\mathbf{v},\psi}^n = \prod_i \psi(v_i)^{n_i} = \psi(n_1 v_1 + \cdots + n_r v_r)$. Recall

$$\zeta(s, A_{\mathbf{v}}, x_\alpha, \chi_{\mathbf{v},\psi}) = \sum_{n \in \mathbb{Z}_+^r} \chi_{\mathbf{v},\psi}^n \prod_j (\alpha^{\sigma_j} + n_1 v_1^{\sigma_j} + \cdots + n_r v_r^{\sigma_j})^{-s_j}.$$

By the lemma, we have

$$\begin{aligned}\sum_{\psi \neq 1} \sum_{\alpha \in \overline{C}_{\mathfrak{a}}(\mathbf{v})} \lambda(\alpha) \zeta(s, A_{\mathbf{v}}, x_\alpha, \chi_{\mathbf{v},\psi}) &\\ = -\sum_{\alpha \in \overline{C}_{\mathfrak{a}}(\mathbf{v})} \lambda(\alpha) \zeta(s, A_{\mathbf{v}}, x_\alpha, \mathbf{1}) &\\ + l \sum_{\beta \in \overline{C}_{\mathfrak{a}\mathfrak{l}}(\mathbf{v})} \lambda(\beta) \zeta(s, A_{\mathbf{v}}, x_\beta, \mathbf{1})&\end{aligned}$$

By choosing l sufficiently large, we may assume $v_i \notin \mathfrak{l}\mathfrak{a}$; i.e., $\psi(v_i) \neq 1$ for all $\psi \neq 1$.

§3.14. Conclusion.

Multiplying by $N(\mathfrak{a})^s\lambda(\mathfrak{a})^{-1}$ and summing over $\mathbf{v}$ and $\mathfrak{a}$, we get

$$\begin{aligned}
\sum_{\mathfrak{a}} \frac{N(\mathfrak{a})^s}{\lambda(\mathfrak{a})} \sum_{\mathbf{v}} \sum_{\psi\neq 1} \sum_{\alpha\in\overline{C}_{\mathfrak{a}}(\mathbf{v})} & \lambda(\alpha)\zeta(s, A_{\mathbf{v}}, x_\alpha, \chi_\psi) \\
&= -\sum_{\mathfrak{a},\mathbf{v}} \frac{N(\mathfrak{a})^s}{\lambda(\mathfrak{a})} \sum_{\alpha\in\overline{C}_{\mathfrak{a}}(\mathbf{v})} \lambda(\alpha)\zeta(s, A_{\mathbf{v}}, x_\alpha, \mathbf{1}) \\
&\quad + l \sum_{\mathfrak{a},\mathbf{v}} \frac{N(\mathfrak{a})^s}{\lambda(\mathfrak{a})} \sum_{\beta\in\overline{C}_{\mathfrak{a}\mathfrak{l}}(\mathbf{v})} \lambda(\beta)\zeta(s, A_{\mathbf{v}}, x_\beta, \mathbf{1}) \\
&= -\sum_{\mathfrak{a},\mathbf{v}} \frac{N(\mathfrak{a})^s}{\lambda(\mathfrak{a})} \sum_{\alpha\in\overline{C}_{\mathfrak{a}}(\mathbf{v})} \lambda(\alpha)\zeta(s, A_{\mathbf{v}}, x_\alpha, \mathbf{1}) \\
&\quad + N(\mathfrak{l}) \frac{\lambda(\mathfrak{l})}{N(\mathfrak{l})^s} \sum_{\mathfrak{a},\mathbf{v}} \frac{N(\mathfrak{l}\mathfrak{a})^s}{\lambda(\mathfrak{l}\mathfrak{a})} \sum_{\beta\in\overline{C}_{\mathfrak{a}\mathfrak{l}}(\mathbf{v})} \lambda(\beta)\zeta(s, A_{\mathbf{v}}, x_\beta, \mathbf{1}) \\
&\overset{(*)}{=} -(1-\lambda(\mathfrak{l})N(\mathfrak{l})^{1-s})L(s,\lambda).
\end{aligned}$$

The identity at $(*)$ is because $\{\mathfrak{l}\mathfrak{a}\}_{\mathfrak{a}} \cong Cl_F^+$.

§3.15. If χ is non-trivial, no variable change necessary. Recall that $F(z, A, x, \chi) = \prod_{i=1}^{r} \frac{\exp(-x_i L_i(z))}{1-\chi_i \exp(-L_i(z))}$ has pole at $L_i(z) = \log \chi + 2\pi i\mathbb{Z}$. If $\chi_i = |\chi_i| e^{i\theta_i}$ with $0 \neq \theta_i \in (-\pi, \pi)$ or $|\chi| < 1$, the pole avoid $z = 0$. Thus the poles avoid original contour $P(\varepsilon)^r$ in z-space. Therefore by the above trick, even to make analytic continuation of $L(s, \lambda)$, Shintani's variable change is not necessary (as long as $\chi_i \neq 1$ for all i).

The corresponding rational function are therefore, writing $t_i = \exp(-L_i(z))$, of the form

$$\prod_{i=1}^{r} \frac{t^{x_i}}{(1 - \psi(v_i) t_i)},$$

where $x_\alpha = (x_1, \ldots, x_r) \in [0, 1]^r \cap \mathbb{Q}^r$.

§3.16. Rationality and integrality theorem. For an embedding $\sigma : F \hookrightarrow \mathbb{R}$, let $\alpha_\sigma \in F^\times$ be an element such that $\alpha_\sigma \equiv 1 (\bmod \mathfrak{n})^\times$ and $\alpha_\sigma^\sigma < 0$ but $\alpha_\sigma^\tau > 0$ for all embedding τ other than σ. The character λ modulo $\mathfrak{n}$ is called totally odd (resp. totally even) if $\lambda(\alpha_\sigma) = -1$ (resp. $\lambda(\alpha_\sigma) = 1$) for all field embeddings σ of F.

Theorem 2 (Siegel 1937, Klingen 1962, Shintani 1976, Pierrete Cassou-Nogues 1979)**.** *Let $F \neq \mathbb{Q}$ be a totally real number field and λ be a finite order ray class character. For $0 < n \in \mathbb{Z}$ and a split prime $\mathfrak{l}$ of F outside $\mathfrak{n}$ with sufficiently large $N(\mathfrak{l})$,*

$$(1 - \lambda(\mathfrak{l})N(\mathfrak{l})^n)L(1-n, \lambda) \in \mathbb{Z}[\lambda],$$

where $\mathbb{Z}[\lambda]$ is the subring of $\mathbb{C}$ generated by the values of λ. We have $L(1-n, \lambda) \neq 0$ only when (i) λ is totally odd and n is odd or (ii) λ is totally even and n is even.

Exercise: Use functional equation to show vanishing of $L(1-n, \lambda)$ when the condition of the theorem is not met.