Lecture note No.2 for Math 205a Fall 2019

Shintani ζ-function

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We extend Hurwitz's method of proving analytic continuation to Shintani ζ-function. For example, consider an imaginary quadratic field $K = \mathbb{Q}[\sqrt{-1}]$ with integer ring $O = \mathbb{Z}[\sqrt{-1}]$. Then $\zeta_K(s) = \sum_{0\neq(\alpha)\subset O} N(\alpha)^{-s}$ for $N(\alpha) = |O/(\alpha)| = |\alpha|^2$. Since $\pm i(a+bi) = \pm ia \mp b$ and $\pm(a+bi) = \pm a \pm bi$, (α) is generated by $a+bi$ o with $a>0$ and $b \geq 0$. Then

$$\zeta_K(s) = \sum_{m>0,n\geq 0}(m+ni)^{-s}(m-ni)^{-s}$$
$$= \zeta(s,1) + \sum_{m=0,n=0}^{\infty} L_1^*((m,n)+(1,1))^{-s}L_2^*((m,n)+(1,1))^{-s}$$
$$= \zeta(s,1) + \zeta((s,s),A,(1,1),(1,1)),$$

where $L_1^*((x,y)) = x+iy$ and $L_2^*((x,y)) = x-iy$ are linear forms with coefficients summarized in a matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$ so that $(x,y)A = (L_1, L_2)$ and $z^s = |z|^s e^{i\theta}$ with $-\pi \leq \theta < \pi$.

§2.1. Set up for Shintani zeta function.

We take a different branch of complex logarithm; so, writing $z = |z|e^{i\theta}$ with $-\pi \leq \theta < \pi$, we define $\log(z) = \log|z| + i\theta$ and hence $z^s = e^{s\log(z)} = |z|^s e^{i\theta s}$.

The data defining Shintani zeta function is

$$A = (a_{ij}) \in M_{r,m}(\mathbb{C}) \quad \text{with } \mathrm{Re}(a_{ij}) > 0 \text{ for all } (i,j),$$
$$\chi = (\chi_1, \ldots, \chi_r) \in \mathbb{C}^r \quad \text{with} \quad |\chi_i| = 1 \quad \text{and}$$
$$x = (x_1, \ldots, x_r) \in [0,1]^r \quad \text{but } x \neq (0,0,\ldots,0).$$

We define Linear form $L_j^* : \mathbb{C}^r \to \mathbb{C}$ by $L_j^*(w) = \sum_{k=1}^r a_{kj} w_k$ for $w = (w_1, \ldots, w_r)$, and Shintani zeta function by

$$\zeta(s, A, x, \chi) := \sum_{n \in \mathbb{N}^r} \chi^n L^*(n+x)^{-s} \quad (s = (s_1, \ldots, s_m) \in \mathbb{C}^m),$$

where $L^*(n+x) = (L_1^*(n+x), \ldots, L_m^*(n+x))$ and $L^*(n+x)^{-s} = \prod_{j=1}^m L_j^*(n+x)^{-s_j}$. Here $\mathbb{N} := \{n \in \mathbb{Z} | n \geq 0\}$.

§**2.2. Convergence Lemma.** *The zeta function* $\zeta(s, A, x, \chi)$ *converges absolutely and locally unifromly if* $\mathrm{Re}(s_i) > \frac{r}{m}$.
Sketch of Proof: Since $|\chi_i| \leq 1$, we may assume that $\chi_i = 1$ for all i; so, $\chi = \mathbf{1} = (1, \ldots, 1)$. Since $\mathrm{Re}(a_{ij}) > 0$, have $\min_{i,j} \mathrm{Re}(a_{ij}) = c > 0$. Assume $\mathrm{Re}(s_j) > 0$ for all j and let $\sigma_j = \mathrm{Re}(s_j)$. To estimate $|\zeta(s, A, x, \mathbf{1})|$, we may throw away finitely many terms; so, we may assume that $n_j > 0$ for all j, and we may ignore x. We have $|L_j^*(n)| = |\,\mathrm{Re}(L_j^*(n)) + \mathrm{Im}(L_j^*(n))| \geq |\,\mathrm{Re}(L_j^*(n))|$; or equivalently,

$$|L_j^*(n)|^{-t} \leq |\,\mathrm{Re}(L_j^*(n))|^{-t} \quad \text{for } t > 0.$$

Thus we may assume that $0 < c = a_{ij} \in \mathbb{R}$ for all (i, j). Note that $\#P_k \leq Ck^{r-1}$ for $P_k := \{k_1 + k_2 + \cdots + k_r = k | 0 < k_j \in \mathbb{Z}\}$ for a constant $C > 0$ independent of $0 < k \in \mathbb{Z}$. Thus $|L_j^*(n)^{-s_j}| \geq (c(\sum_{i=1}^r n_i))^{-\sigma_j}$ and hence, writing $|\sigma| = \sum_{j=1}^m \sigma_j$,

$$|\zeta(s, A, x, \chi)| \leq c^{-|\sigma|} \sum_{k=1}^{\infty} \sum_{n \in P_k} k^{-|\sigma|} \leq c^{-|\sigma|} C \sum_{k=1}^{\infty} k^{r-1-|\sigma|},$$

which converges absolutely and locally uniformly if $|\sigma| > r$. $\square$

§2.3. Rational function in $\exp(-L_i(t))$.

Define $G(t) = G(t, A, x, \chi) := \sum_{n \in \mathbb{N}^r} \chi^n \exp(-\sum_{j=1}^m L_j^*(n+x)t_j)$. Note that

$$\sum_{j=1}^m L_j^*(n+x)t_j = \sum_{j=1}^m \sum_{i=1}^r a_{ij}(n_i + x_i)t_j = \sum_{i=1}^r (n_i + x_i) \sum_{j=1}^m a_{ij}t_j.$$

Thus writing $L_i(t) := \sum_{j=1}^m a_{ij}t_j$ for $t = (t_1, \ldots, t_m)$, we have

$$\begin{aligned} G(t) = G(t, A, x, \chi) &= \prod_{i=1}^r \sum_{n_i=0}^\infty \chi_i^{n_i} \exp(-(n_i + x_i)L_i(t)) \\ &= \exp(-\sum_{i=1}^r x_i L_i(t)) \prod_{i=1}^r \sum_{n_i=0}^\infty \chi_i^{n_i} \exp(-n_i L_i(t)) \\ &= \prod_{i=1}^r \frac{\exp(-x_i L_i(t))}{1 - \chi_i \exp(-L_i(t))}. \end{aligned}$$

Let $\widetilde{G}(t)$ be the function constructed for $\chi = \mathbf{1}$, $\mathrm{Re}(A)$ and $x = \mathbf{1}$. Thus $|\widetilde{G}(t)| = O(\exp(-c\sum_i t_i))$ as $|t| \to \infty$ and is bounded as $|t| \to 0$; i.e., $|\widetilde{G}(t)|$ is an integrable positive valued function over $\mathbb{R}_+^r$.

§2.4. Integral expression of $\zeta(s, A, \chi, x)$.

For $G_N(t) = \sum_{|n|<N} \chi^n \exp(-\sum_{j=1}^m L_j^*(n+x)t_j)$, $|G_N(t)t^s| \leq |\widetilde{G}(t)|t^\sigma$ for all $0 < N \in \mathbb{Z}$, by Lebesgue's dominated convergence theorem, we find for $d\mu = \prod_j t_j^{-1} dt_j$, if $\mathrm{Re}(s_j) > m/r$ for all j,

$$\begin{aligned}\Gamma_m(s)\zeta(s, A, x, \chi) &= \sum_{n\in\mathbb{N}^r} \int_0^\infty \cdots \int_0^\infty t^s \chi^n \exp(-\sum_{j=1}^m L_j^*(n+x)t_j) d\mu \\ &= \lim_{N\to\infty} \int_0^\infty \cdots \int_0^\infty t^s G_N(t) d\mu \\ &= \int_0^\infty \cdots \int_0^\infty t^s \lim_{N\to\infty} G_N(t) d\mu = \int_0^\infty \cdots \int_0^\infty t^s G(t) d\mu,\end{aligned}$$

where $\Gamma_m(s) = \prod_{j=1}^m \Gamma(s_j)$. Let $\mathbb{R}_+ := \{x \in \mathbb{R} | x > 0\}$ (right half real line), and hereafter we write

$$\int_0^\infty \cdots \int_0^\infty ? d\mu = \int_{\mathbb{R}_+^m} ? d\mu.$$

§2.5. $\int_{P(\varepsilon)} \cdots \int_{P(\varepsilon)} t^s G(t) d\mu$ does not make sense.

For simplicity, take $\chi = \mathbf{1}$. The function $\frac{\exp(-x_i L_i(z))}{1-\exp(-L_i(z))}$ has pole on the hyper plane $H_{i,k}$ defined by $L_i(z) = 2k\pi\sqrt{-1}$ for any $k \in \mathbb{Z}$. Plainly $P(\varepsilon)^r$ intersects some of $H_{i,k}$; so, taking naive contour integral does not make sense.

Shintani's idea is to make a very clever variable change. Decompose

$$\mathbb{R}_+^m = \bigcup_{k=1}^m D_k \quad \text{with} \quad D_k = \{(t_1, \ldots, t_m) | t_k \geq t_i \ \forall i \neq k\}.$$

Note that $D_k \cap D_j \subset \{t_k = t_j\}$; so, the intersection of D_k and D_j for $k \neq j$ has measure 0. Any $t = (t_1, \ldots, t_m) \in \mathbb{R}_+^m$, taking the index k such that $t_k = \max_j(t)$, we find $t \in D_k$; so, $\mathbb{R}_+^m$ is a union of D_k. Thus we have for any continuous integrable function $f(t)$ on $\mathbb{R}_+^m$, we have $\int_{\mathbb{R}_+^m} f(t) d\mu = \sum_{k=1}^m \int_{D_k} f(t) d\mu$. Thus we study $\int_{D_k} t^s G(t) d\mu$.

§**2.6. Shintani's variable change.** Since $D_k \cong D_1$ by interchanging t_k and t_1, we study the integral over D_1. The variable change is $(t_1,\dots,t_m) = U(1,u_2\dots,u_{m-1},u_m)$ (i.e., on D_k, $u_k = 1$). Thus $U \in \mathbb{R}_+$ and $u_j \in (0,1]$. The jacobian matrix of this variable change is given

$$\begin{pmatrix} \frac{\partial t_1}{\partial U} & \frac{\partial t_2}{\partial U} & \cdots & \frac{\partial t_m}{\partial U} \\ \frac{\partial t_1}{\partial u_2} & \frac{\partial t_2}{\partial u_2} & \cdots & \frac{\partial t_m}{\partial u_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial t_1}{\partial u_m} & \frac{\partial t_2}{\partial u_m} & \cdots & \frac{\partial t_m}{\partial u_m} \end{pmatrix} = \begin{pmatrix} 1 & u_2 & \cdots & u_m \\ 0 & U & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U \end{pmatrix}.$$

So we have $dt = U^{m-1}dU\,du$ for $du = du_2\cdots du_m$ and

$$d\mu = U^{-1}u^{-1}dU\,du \quad (\mathrm{Tr}(s) = \sum_{j=1}^{m} s_j) \text{ and}$$

$$\int_{D_1} t^s G(t)d\mu = \int_0^\infty \int_0^1 \cdots \int_0^1 G_1(U,u)U^{\mathrm{Tr}(s)}u^s d\mu \text{ with}$$

$$G_1(U,u) = \prod_{i=1}^{r} \frac{\exp(-x_i U L_i(u))}{1-\chi_i \exp(-U L_i(u))}.$$

§2.7. Why the contour $P(\varepsilon)\times P(\varepsilon,1)^{m-1}$ avoids singularity?

Write $\chi=|\chi_i|e^{i\theta_i}$ with $0\le\theta_i<2\pi$. Since $u=(1,u_2,\dots,u_m)\in(0,1]^m$,

$$\mathrm{Re}(L_i(u))=\mathrm{Re}(a_{i1})+\sum_{j=2}\mathrm{Re}(a_{ij})u_j\to\mathrm{Re}(a_{i1})\ge c$$

as $u_j\to 0$ ($j=2,\dots m$). Thus $|u_j|\le\varepsilon$ for sufficiently small ε, $1-\chi_i\exp(-UL_i(u))$ have 0 only at $U=0$ if $\chi=1$ and otherwise $UL_i(u)\in\theta_i+2\pi i\mathbb{Z}+\log|\chi_i|$ which never happen if ε is sufficiently small. This argument applies to all integral over D_k for $k>1$, and making ε further small and writing $G_k(U,u)=\prod_{i=1}^r\frac{\exp(-x_iUL_i(u))}{1-\chi_i\exp(-UL_i(u))}$ for the function on D_k, we get

$$\begin{aligned}&\Gamma_m(s)\zeta(s,A,x,\chi)\\&\quad=\frac{\sum_{k=1}^m(e^{2\pi s_k i}-1)\int_{P(\varepsilon)}\int_{P(\varepsilon,1)}\cdots\int_{P(\varepsilon,1)}G_k(U,u)U^{\mathrm{Tr}(s)}u^s d\mu}{(e^{2\pi i\mathrm{Tr}(s)}-1)\prod_{j=1}^m(e^{2\pi s_j i}-1)}.\end{aligned}$$

Thus Shintani zeta function is analytically continued to a meromorphic function on $s\in\mathbb{C}^m$ with possible poles at $\mathrm{Tr}(s)\in\mathbb{Z}$ and $s_j\in\mathbb{Z}$.

§2.8. Generalized Bernoulli polynomial.

Let $\mathbb{Q}(A,\chi)$ be the field generated by coefficients of A and χ inside $\mathbb{C}$. Assume $\chi_i \neq 1$ for all i. Then $G(z)$ is holomorphic at $z = 0$; so, we have its Taylor expansion around $z = 0$:

$$G(z,A,x,\chi) = \prod_{i=1}^{r} \frac{\exp(-x_i L_i(z))}{1 - \chi_i \exp(-L_i(z))} = \sum_{n\in\mathbb{N}^m} \frac{B_{n+\mathbf{1}}(x)}{(n+\mathbf{1})!} z^n$$

for the polynomial $B_{n+\mathbf{1}}(x) \in \mathbb{Q}(A,\chi)[x]$ of r variables, where $n! = \prod_{j=1}^{m} n_j!$, $z^n = \prod_{j=1}^{m} z_j^{n_j}$. Similarly, writing $(Uu)^n$ for z^n under the variable change $z = U(u_1,\dots,u_{k-1},1,u_{k+1},\dots,u_m)$,

$$G_k(U,u,A,x,\chi) = \prod_{i=1}^{r} \frac{\exp(-x_i U L_i(u))}{1 - \chi_i \exp(-U L_i(u))} = \sum_{n\in\mathbb{N}^m} \frac{B^{(k)}_{n+\mathbf{1}}(x)}{(n+\mathbf{1})!} (Uu)^n$$

$$\text{and} \quad B_n(x) = \sum_{k=1}^{m} B_n^{(k)}(x).$$

§**2.9. Rationality Theorem.** As we have seen, for $0 < n \in \mathbb{Z}$,

$$\left[(e^{2\pi i m s}-1)\Gamma(s)\left((e^{2\pi i s}-1)\Gamma(s)\right)^{m-1}\right]_{s=1-n}\zeta(1-n,A,x,\chi)$$
$$=\sum_{k=1}^{m}\oint\oint\cdots\oint G_k(U,u)U^{m(1-n)}u^{1-n}d\mu,$$

$$\left[(e^{2\pi i m s}-1)\Gamma(s)\left((e^{2\pi i s}-1)\Gamma(s)\right)^{m-1}\right]_{s=1-n}=m\left(\frac{(-1)^{n-1}2\pi i}{(n-1)!}\right)^{m}.$$

The coefficient in $(Uu)^{-\mathbf{1}}$ of $G(z)(Uu)^{-n\mathbf{1}}$ is given by the coefficient in $(Uu)^{(n-1)\mathbf{1}}$; so, if $\boxed{\chi_i \neq 1}$ for all i,

$$m\left(\frac{(-1)^{n-1}2\pi i}{(n-1)!}\right)^{m}\zeta((1-n)\mathbf{1},A,x,\chi)$$
$$=(2\pi i)^m\sum_{k=1}^{m}\mathrm{Res}_{U=0,u=0}(G_k(U,u)U^{m(1-n)}u^{1-n})=(2\pi i)^m\frac{B_n(x)}{n!^m}$$

$$\text{and}\quad \zeta((1-n)\mathbf{1},A,x,\chi)=(-1)^{m(n-1)}\frac{B_n(x)}{mn^m}\in\mathbb{Q}(A,\chi)[x].$$

§2.10. A bit of algebraic number theory.

Let F be a number field with integer ring O. A non-zero finitely generated O-submodule $\mathfrak{a}$ of F is called a fractional ideal of F (in particular, a fractional ideal is a finitely generated $\mathbb{Z}$-module). Let I be the correction of all fractional ideals of F. The trace pairing $(x, y) \mapsto \mathrm{Tr}(xy)$ gives a non-degenerate self $\mathbb{Q}$-duality on F with $(\alpha x, y) = (x, \alpha y)$ for $\alpha \in O$. For a fractional ideal $\mathfrak{a}$, $\mathfrak{a}^* := \{x \in F | (O, x) \subset \mathbb{Z}\}$, which is a fractional ideal as it is finitely generated even over $\mathbb{Z}$. Define $\mathfrak{a}^{-1} = \{x \in F | x\mathfrak{a} \subset O\}$, which is an O-module (as $x\mathfrak{a} \subset O \Rightarrow Ox\mathfrak{a} \subset O$). Plainly $\mathfrak{a}^{-1} \subset \mathfrak{a}^*$; so, $\mathfrak{a}^{-1}$ is finitely generated over $\mathbb{Z}$; so, $\mathfrak{a}^{-1}$ is a fractional ideal. Defining $\mathfrak{ab}$ be the O-module generated by ab with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$, we have $\mathfrak{a}^{-1}\mathfrak{a} = O$. Thus I is a group by the above product $(\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{ab}$ with identity O.

§**2.11. Ray class group.** Since O is a Dedekind domain, we can uniquely factorize $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{e(\mathfrak{p})}$ for prime ideals $\mathfrak{p}$. The exponent $e(\mathfrak{p})$ is non-negative if $\mathfrak{a}$ is an O-ideal. Thus $I \cong \bigoplus_{\mathfrak{p}:\text{prime}} \mathfrak{p}^{\mathbb{Z}}$. Pick a non-zero O-ideal $\mathfrak{n}$. If in the above factorization, any prime in $\mathfrak{n}$ does not appear in $\mathfrak{a}$, we say that $\mathfrak{a}$ is prime to $\mathfrak{n}$. Then plainly $I(\mathfrak{n})$ made up of ideals prime to $\mathfrak{n}$ is a subgroup of I.

We write $\alpha \equiv 1 \pmod{\mathfrak{n}}^{\times}$ if $\alpha = \frac{a}{b}$ with $a, b \in O$ such that $a \equiv b \equiv 1 \mod \mathfrak{n}$. If in addition we impose a/b to be positive under every real embedding of F (i.e., totally positive), we write $\alpha \equiv 1 \pmod{\mathfrak{n}\infty}^{\times}$. Then we define $P(\mathfrak{n}) = \{(\alpha) | \alpha \equiv 1 \pmod{\mathfrak{n}}^{\times}\}$ and $P^{+}(\mathfrak{n}) = \{(\alpha) | \alpha \equiv 1 \pmod{\mathfrak{n}\infty}^{\times}\}$

$$Cl_F(\mathfrak{n}) := \frac{I(\mathfrak{n})}{P(\mathfrak{n})} \quad \text{and} \quad Cl_F^{+}(\mathfrak{n}) := \frac{I(\mathfrak{n})}{P^{+}(\mathfrak{n})},$$

which are finite group. The group $Cl_F(\mathfrak{n})$ (resp. $Cl_F^{+}(\mathfrak{n})$) is called the (resp. strict) ray class group modulo $\mathfrak{n}$.

§2.12. Examples.

If $F = \mathbb{Q}$, we have $\mathfrak{n} = (N)$ for a positive integer N. To gete a map from $(\mathbb{Z}/N\mathbb{Z})^\times$ into the class grup, we send $0 \neq n \in \mathbb{Z}$ into the class $[(n)]$ of the principal ideal (n) in $Cl^+_{\mathbb{Q}}(N)$. If $[(n)] = [(n')]$, then $n' = \frac{n'}{n} n$ with $\frac{n'}{n} \equiv 1 (\text{mod } N\infty)^\times$ ($\Leftrightarrow n' \equiv n$ mod N and $nn' > 0$). Thus $(\mathbb{Z}/N\mathbb{Z})^\times \hookrightarrow Cl^+_{\mathbb{Q}}(N)$. Take $0 \neq \alpha = \frac{a}{b}$ (reduced fraction) with a, b prime to N. Choose an integer $n \neq 0$ so that modulo N, $[a \mod N][b \mod N]^{-1} = [n \mod N]$ and $\alpha n > 0$. Then (α) is in the image of $(\mathbb{Z}/N\mathbb{Z})^\times$. This shows

$$(\mathbb{Z}/N\mathbb{Z})^\times \cong Cl^+_{\mathbb{Q}}(N).$$

In general, writing $O^\times_+$ (resp. $\overline{O}^\times_+$) be the group of totally positive units in O (resp. its image in $(O/\mathfrak{n})^\times$). Then we have an exact sequence for $i(\alpha) = [(\alpha)]$

$$1 \to (O/\mathfrak{n}O)^\times/\overline{O}^\times_+ \xrightarrow{i} Cl^+_F(\mathfrak{n}) \to Cl^+_F \to 1.$$

For a divisor $\mathfrak{d} \supset \mathfrak{n}$, we have a projection $Cl^?_F(\mathfrak{n}) \twoheadrightarrow Cl^?_F(\mathfrak{d})$.

§2.13. Hecke L-function.

Consider a character $\lambda : Cl_F^+(\mathfrak{n}) \to \mathbb{C}^\times$. Pull back λ to a character of $I(\mathfrak{n})$ and extend it putting $\lambda(\mathfrak{a}) = 0$ for fractional ideals $\mathfrak{a}$ with non-trivial common divisor with $\mathfrak{n}$. Define a Hecke L-function

$$L(s,\lambda) = \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} (1 - \lambda(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1}.$$

As we will see that this converges absolutely and locally uniformly if Re$(s) > 1$. A weight for F is a formal integer linear combination $k = \sum_\sigma k_\sigma \sigma$ of complex embeddings σ of F. For $\alpha \in F$, we write $\alpha^k = \prod_\sigma \alpha^{k_\sigma \sigma}$. We say k admissible if $\varepsilon^k = 1$ for all $\varepsilon \in O_+^\times$. A character $\lambda : I(\mathfrak{n}) \to \mathbb{C}^\times$ is called a Hecke character of (admissible) weight k if $\lambda(\alpha) = \alpha^k$ for all $(\alpha) \in P^+(\mathfrak{n})$. In this case, the character λ can be of infinite order. We then define

$$L(s,\lambda) = \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} (1 - \lambda(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1},$$

which is again absolutely and locally uniformly convergent if Re$(s) \gg 0$. For $\mathbb{Q}$ or an imaginary quadratic field, every weight is admissible.

§**2.14. Real quadratic field.** Let $F=\mathbb{Q}[\sqrt{D}]$ with $0<D\in\mathbb{Z}$. For simplicity, assume $|Cl_F^+|=1$. By Dirichlet unit theorem (or by the solution of Pell's equation), we have a fundamental unit $\varepsilon\in O_+^\times$ such that $O_+^\times=\varepsilon^{\mathbb{Z}}$. Embed F into $\mathbb{R}\times\mathbb{R}$ by $\alpha\mapsto(\alpha,\sigma(\alpha))$, where $\sigma(a+b\sqrt{D})=a-b\sqrt{D}$. We regard F as a $\mathbb{Q}$ vector subspace of $\mathbb{R}^2$ in this way. Let $F_+^\times=F\cap(\mathbb{R}_+^\times\times\mathbb{R}_+^\times)$. Let $\mathbf{1}=(1,1)$ and $\varepsilon=(\varepsilon,\sigma(\varepsilon))$ and A be the 2×2 matrix with row vectirs given by $\mathbf{1}$ and ε. Let $C=C(\mathbf{1},\varepsilon)=\mathbb{R}_+^\times\mathbf{1}+\mathbb{R}_+\varepsilon$ (a cone generated by $\mathbf{1},\varepsilon$) for $\mathbb{R}_+=\mathbb{R}_+^\times\cup\{0\}$. By multiplying ε, we can bring any $0\neq\alpha\in O$ inside C uniquely. Note that $R=O\cap((0,1]\mathbf{1}+[0,1]\varepsilon)$ is finite, and we regard $R\subset\mathbb{R}^2$ by $(0,1]\times[0,1]\ni x=(x_1,x_2)\mapsto x_1\mathbf{1}+x_2\varepsilon$. Then we have an absolutely and uniformly convergent series for $\mathrm{Re}(s)>1$ $(m=r)$,

$$\zeta_F(s)=\sum_{x\in R}\zeta((s,s),A,\mathbf{1},x)$$

which has analytic continuation for all $s\in\mathbb{C}$. We are going to generalize this to number fields F and any Hecke character λ.