Lecture note No.1 for Math 205a Fall 2019

Dirichlet L-function

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We extend Hurwitz's method of proving analytic continuation and functional equation to Dirichlet L-function. I suggest you to try working out Riemann's method yourself. Take an integer $N \geq 1$. For a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, Dirichlet L-function with character χ is defined by $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$ which is absolutely convergent in the region $\mathrm{Re}(s) > 1$. Here if n has a non-trivial common factor with N, we agree to set $\chi(n) = 0$ as a convention. The residue ring $\mathbb{Z}/N\mathbb{Z}$ surjects down to $\mathbb{Z}/D\mathbb{Z}$ for any divisor D of N; so, any character $\chi_0 : (\mathbb{Z}/D\mathbb{Z})^\times \to \mathbb{C}^\times$ can be pulled back to a character of $(\mathbb{Z}/N\mathbb{Z})^\times$. If there is no divisor D of N such that χ comes from $(\mathbb{Z}/D\mathbb{Z})^\times$ in this way, χ is called *primitive*.

§1.1. Hurwitz zeta function.

For $0 < x \leq 1$, we define

$$\zeta(s,x) := \sum_{n=0}^{\infty} (n+x)^{-s},$$

which is absolutely and locally uniformly convergent if $\mathrm{Re}(s) > 1$. Indeed, if $\sigma = \mathrm{Re}(s) > 1$, we have

$$|\zeta(s,x)| \leq \sum_{n=1}^{\infty} n^{-\sigma} \leq 1+\int_1^{\infty} x^{-\sigma} dx = 1+\left[\frac{x^{-\sigma+1}}{-\sigma+1}\right]_1^{\infty} = 1+(\sigma-1)^{-1}.$$

Note that

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \sum_{a=0}^{N} \chi(a)(Nn+a)^{-s} = \sum_{a=0}^{N} \chi(a) N^{-s} \zeta(s, \frac{a}{N}),$$

which is also absolutely and locally uniformly convergent if $\mathrm{Re}(s) > 1$. Thus $\zeta(s,x)$ and $L(s,\chi)$ are holomorphic function in the right half plane $\{s \in \mathbb{C} | \mathrm{Re}(s) > 1\}$.

§**1.2. Contour integral.** Define $G(t,x) := \frac{e^{(1-x)t}}{e^t-1} = \frac{e^{-tx}}{1-e^{-t}} \overset{(*)}{=} \sum_{n=0}^{\infty} e^{-(n+x)t}$. The identity $(*)$ is valid if $\mathrm{Re}(t) > 0$. Note that $G(t,x) = O(e^{-tx})$ as $t \to \infty$ and $G(t,x) = O(t^{-1})$ as $t \to 0$ as the Laurent expansion of $G(t,x)$ starts with t^{-1}. Hence the Mellin transform $\int_0^\infty t^{s-1} G(t,x)dt$ converges if $\mathrm{Re}(s) > 1$ and we get

$$\int_0^\infty t^{s-1} G(t,x)dt = \Gamma(s)\zeta(s,x).$$

Therefore allowing variable $z = re^{i\theta}$ $0 \le \theta \le 2\pi \in \mathbb{C}$

$$\begin{aligned}\int_{P(\varepsilon)} z^{s-1} G(z,x)dz = -\int_\varepsilon^\infty t^{s-1} G(t,x)dt + e^{2\pi i s} \int_\varepsilon^\infty t^{s-1} G(t,x) \\ + \oint_{|z|=\varepsilon} z^{s-1} G(z,x)dz\end{aligned}$$

has limit as $\varepsilon \to 0$ when $\mathrm{Re}(s) > 1$ given by

$$\lim_{\varepsilon \to 0} \int_{P(\varepsilon)} z^{s-1} G(z,x)dz = (e^{2\pi i s} - 1)\Gamma(s)\zeta(s,x) \quad \text{if } \mathrm{Re}(s) > 1.$$

§1.3. Meromorphic continuation.

By residue theorem, for $0 < \varepsilon' < \varepsilon < 1$, we have for the annulus $A(\varepsilon', \varepsilon)$ with inner radius ε' and outer radius ε

$$\int_{P(\varepsilon)} z^{s-1}G(z,x)dz - \int_{P(\varepsilon')} z^{s-1}G(z,x)dz = \int_{A(\varepsilon',\varepsilon)} G(z,x)dz = 0,$$

and hence $s \mapsto \int_{P(\varepsilon)} z^{s-1}G(z,x)dz$ is well defined for all $s \in \mathbb{C}$ giving the identity $(e^{2\pi is} - 1)\Gamma(s)\zeta(s,x) = \int_{P(\varepsilon)} z^{s-1}G(z,x)dz$ which is an entire function of $s \in \mathbb{C}$. Therefore

$$\zeta(s,x) = \frac{\int_{P(\varepsilon)} z^{s-1}G(z,x)dz}{(e^{2\pi is} - 1)\Gamma(s)}$$

is a meromorphic function on $\mathbb{C}$ with possible poles at integers $n \leq 1$. Since $\Gamma(s)^{-1} = \frac{(s+n-1)(s+n-2)\cdots s}{\Gamma(s+n)}$, $\Gamma(s)^{-1}$ has simple zero at $-n$ ($0 \geq n \in \mathbb{Z}$). Thus the poles of $\zeta(s,x)$ is limited to $s = 1$. Similarly we get $\Gamma(s) = \frac{\int_{P(\varepsilon)} z^{s-1}e^{-z}dz}{(e^{2\pi is}-1)}$; so, $\Gamma(s)$ has simple pole at $s = 1 - n$ ($0 < n \in \mathbb{Z}$) with residue $\frac{(-1)^{n-1}(2\pi i)}{(n-1)!}$.

Exercise: Prove the above residue formula for $\Gamma(s)$.

§1.4. Functional equation of $\zeta(s,x)$.

The function $G(z,x)$ has simple pole at $2\pi in$ for integers n and

$$\mathrm{Res}_{z=2\pi in} G(z,x)z^{s-1} = \begin{cases} e^{2\pi nx+(s-1)\pi i/2}|2n\pi|^{s-1} & \text{if } n>0, \\ e^{-2\pi nx+3(s-1)\pi i/2}|2n\pi|^{s-1} & \text{if } n<0 \end{cases}$$

This shows

$$\int_{D(m)} G(z,x)z^{s-1}dz$$
$$= (2\pi)^s\left(e^{3\pi is/2}\sum_{n=1}^{m} e^{2\pi inx}n^{s-1} - e^{\pi is/2}\sum_{n=1}^{m} e^{-2\pi inx}n^{s-1}\right)$$

Assuming $\mathrm{Re}(s)<0$ and passing to the limit, we get

$$(e^{2\pi is}-1)\Gamma(s)\zeta(s,x) = \int_{P(\varepsilon)} G(z,x)z^{s-1}dz = \lim_{m\to\infty}\int_{D(m)} G(z,x)z^{s-1}dz$$
$$= (2\pi)^s\left(e^{3\pi is/2}\sum_{n=1}^{\infty} e^{2\pi inx}n^{s-1} - e^{\pi is/2}\sum_{n=1}^{\infty} e^{-2\pi inx}n^{s-1}\right).$$

§1.5. Towards the functional equation of $L(s,\chi)$.

Plug in the functional equation of $\zeta(s,x)$ into

$$L(s,\chi)=\sum_{a=0}^{N}\chi(a)N^{-s}\zeta(s,\frac{a}{N}),$$

we get

$$(e^{2\pi is}-1)\Gamma(s)L(s,\chi)=$$

$$\Big(\frac{2\pi}{N}\Big)^{s}(e^{3\pi is/2}\sum_{n=1}^{\infty}\sum_{a=0}^{N}\chi(a)\mathbf{e}(\frac{na}{N})n^{s-1}-e^{\pi is/2}\sum_{n=1}^{\infty}\sum_{a=1}^{N-1}\chi(a)\mathbf{e}(\frac{-na}{N})n^{s-1}).$$

Let $G(\chi)=\sum_{a=1}^{N-1}\chi(a)\mathbf{e}(\frac{a}{N})$. Then by $na\mapsto a$ (if $a\in(\mathbb{Z}/N\mathbb{Z})^{\times}$, we get $\sum_{a=1}^{N-1}\chi(a)\mathbf{e}(\frac{na}{N})=\chi(n)^{-1}G(\chi)$. Assuming χ primitive, we will prove in the next page

$$\sum_{a=1}^{N-1}\chi(a)\mathbf{e}(\frac{na}{N})=0=\chi^{-1}(n)G(\chi)\ \text{ if } (n,N):=\text{GCD of } n \text{ and } N\neq 1.$$

§1.6. Gauss sum for primitive χ.

Lemma 1. *Let G be a finite abelian group and $\chi : G \to \mathbb{C}^\times$ be a character. If χ is non-trivial, $\sum_{g\in G} \chi(g) = 0$.*

Let h be the order of χ. Since the image H of χ is a group μ_h of h-th roots of unity (which is cyclic), we may assume that G is cyclic and $G \cong H$ by χ. Then we have $\sum_{\zeta\in\mu_h} \zeta = \sum_{g\in G} \chi(g)$ which vanishes as it is the trace term of $X^h - 1$. $\square$

Suppose to have a prime $p|n$ with $n = pn'$ and $p|N$ with $N = pD$, and put $\psi(a) = \mathbf{e}(a/N)$. Then

$$\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)\mathbf{e}(na/N) = \sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)\mathbf{e}(na/N)$$

$$= \sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)\mathbf{e}(n'a/D) = \sum_{a\in(\mathbb{Z}/D\mathbb{Z})^\times} \mathbf{e}(n'a/D)\chi(a) \sum_{b\equiv 1 \mod D} \chi(b),$$

which vanishes as χ restricted to $\{b \in (\mathbb{Z}/N\mathbb{Z})^\times | b \equiv 1 \mod D\}$ is nontrivial by primitivity.

§**1.7. Functional equation of $L(s,\chi)$ of Euler type.** By the discussion in §1.6, as long as χ is primitive, we have

$$\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times}\chi(a)\mathbf{e}(na/N)=\chi^{-1}(n)G(\chi)$$

including n not co prime to N (following the convention $\chi^{-1}(n)=0$ for such n). Therefore we obtain, if $\mathrm{Re}(s)<0$,

$$(e^{2\pi is}-1)\Gamma(s)L(s,\chi)=\left(\frac{2\pi}{N}\right)^s G(\chi)(e^{3\pi is/2}-\chi(-1)e^{\pi is/2})L(1-s,\chi).$$

Since we already know the analytic continuation of $L(s,\chi)$, this identity is valid for all $s\in\mathbb{C}$. Since

$$\frac{(e^{2\pi is}-1)}{(e^{3\pi is/2}-\chi(-1)e^{\pi is/2})}=\begin{cases}2\cos(\pi s/2) & \text{if } \chi(-1)=1,\\ 2\sqrt{-1}\sin(\pi s/2) & \text{if } \chi(-1)=-1,\end{cases}$$

we get

$$L(s,\chi)=\begin{cases}\frac{G(\chi)(2\pi/N)^sL(1-s,\chi^{-1})}{2\Gamma(s)\cos(\pi s/2)} & \text{if } \chi(-1)=1,\\ \frac{G(\chi)(2\pi/N)^sL(1-s,\chi^{-1})}{2\sqrt{-1}\Gamma(s)\sin(\pi s/2)} & \text{if } \chi(-1)=-1.\end{cases}$$

§1.8. Functional equation of $L(s,\chi)$ of Riemann type.

Put

$$\widehat{L}(s,\chi) = \begin{cases} \pi^{-s/2}\Gamma(\frac{s}{2})L(s,\chi) & \text{if } \chi(-1)=1, \\ \pi^{-(s+1)/2}\Gamma(\frac{s+1}{2})L(s,\chi) & \text{if } \chi(-1)=-1. \end{cases}$$

By $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$ (Euler) and

$$\frac{\pi^{s-(1/2)}\Gamma((1-s)/2)}{\Gamma(s/2)} = \frac{(2\pi)^s}{2\Gamma(s)\cos(\pi s/2)}$$

we have seen in §0.14 (or by Riemann's method), we can interpret the functional equation in §1.7 into the following form:

$$\widehat{L}(s,\chi) = \kappa(\chi)N^{\frac{1}{2}-s}\widehat{L}(1-s,\chi^{-1})$$

$$\text{with } \kappa(\chi) = \begin{cases} G(\chi)/\sqrt{N} & \text{if } \chi(-1)=1, \\ -\sqrt{-1}G(\chi)/\sqrt{N} & \text{if } \chi(-1)=-1. \end{cases}$$

Thus $\widehat{L}(s,\chi) = \kappa(\chi)\kappa(\chi^{-1})\widehat{L}(s,\chi)$ and hence $|\kappa(\chi)| = 1$.

Exercise: Why $|\kappa(\chi)| = 1$?

§1.9. Bernoulli polynomials.

Let $F(z,x) := zG(z,1-x) = \frac{ze^{zx}}{e^z-1}$ and expand $F(z,x)$ into a power series in z: $F(z,x) = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}$. Since $e^{zx} = \sum_{n=0}^{\infty} \frac{x^n z^n}{n!}$, $B_n(x)$ is a polynomial of x. Note that $F(z,x) = e^{zx}\frac{z}{e^z-1} = \left(\sum_{n=0}^{\infty} x^n \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} B_m \frac{z^m}{m!}\right)$. Thus we get

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j} \quad \text{and}$$

$$\mathrm{Res}_{z=0} G(z,x) z^{-n} = \mathrm{Res}_{z=0} F(z,1-x) z^{-n-1} = \frac{B_n(1-x)}{n!}.$$

Since $F(z,1-x) = \frac{ze^{(1-x)z}}{e^z-1} = \frac{ze^{-xz}}{1-e^{-z}} = F(-z,x)$, we get

$$B_n(1-x) = (-1)^n B_n(x).$$

Examples: $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{x}{6}$, ...

§1.10. The value of $\zeta(s,x)$ at $s = 1-n \le 0$. Let $\mathbb{Q}(\chi)$ be the field inside $\mathbb{C}$ generated by the values of χ. Thus writing h for the order of χ, $\mathbb{Q}(\chi) = \mathbb{Q}(\mu_h)$ is the field of h-th roots of unity. By the formula given in §1.3, we have

$$(e^{2\pi is} - 1)\Gamma(s)\zeta(s,x) = \int_{P(\varepsilon)} z^{s-1}G(z,x)dz.$$

If s is an integer $1-n$ $(0 < n \in \mathbb{Z})$, then $z^{s-1}G(z,x)$ is single valued, and by $(e^{2\pi is} - 1)\Gamma(s)|_{s=1-n} = \frac{(-1)^{n-1}2\pi i}{(n-1)!}$, the formula becomes

$$\frac{(-1)^{n-1}2\pi i}{(n-1)!}\zeta(1-n,x) = \int_{\partial D_\varepsilon} z^{-n}G(z,x)dz$$
$$= 2\pi i \mathrm{Res}_{z=0} z^{-n}G(z,x) = 2\pi i\frac{B_n(1-x)}{n!} = (-1)^n 2\pi i\frac{B_n(x)}{n!}$$

for the circle of radius ε centered at the origin.

§**1.11. Rationality Theorem.** We have, for $0 < n \in \mathbb{Z}$

$$\zeta(1-n,x) = \frac{B_n(x)}{n} \quad \text{and}$$

$$L(1-n,\chi) = -\sum_{b=0}^{N} \chi(b) N^n \frac{B_n(b/N)}{n} \in \mathbb{Q}(\chi).$$

If $\chi(-1) \neq (-1)^n$ and $\chi \neq 1$, $L(1-n,\chi) = 0$, and $\zeta(1-n) = 0$ if $n > 1$ is odd. In the above formula, χ can be any function on $(\mathbb{Z}/N\mathbb{Z})$.

The last assertion follows from

$$L(1-n,\chi) \overset{b \mapsto N-b}{=} -\sum_{b=0}^{N} \chi(-b) N^n \frac{B_n((N-b)/N)}{n}$$

$$= -(-1)^n \chi(-1) \sum_{b=0}^{N} \chi(b) N^n \frac{B_n(b/N)}{n} = (-1)^n \chi(-1) L(1-n,\chi)$$

by $B_n((N-b)/N) = (-1)^n B_n(b/N)$.

§**1.12. Rational function expression.** Put $t = e^{z/N}$. Then

$$\frac{t^{Nx}}{t^N - 1} = \frac{1}{z}F(z,x) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{B_{n+1}(x)}{(n+1)!} z^n$$

and for any integer $a > 1$,

$$\frac{t^{Nx}}{t^N - 1} - a\frac{t^{aNx}}{t^{Na} - 1} = \sum_{n=0}^{\infty} \frac{B_{n+1}(x) - a^{n+1}B_{n+1}(x)}{(n+1)!} z^n$$

and by $dt^N = Nt^{N-1}dt \Leftrightarrow Nt^N\frac{d}{dt^N} = t\frac{d}{dt}$

$$\begin{aligned}(1-a^{n+1})N^n\frac{B_{n+1}(x)}{n+1} &= N^n\left(t^N\frac{d}{dt^N}\right)^n \left[\frac{t^{Nx}}{t^N - 1} - a\frac{t^{aNx}}{t^{Na} - 1}\right]\Bigg|_{t=1} \\ &= \left(t\frac{d}{dt}\right)^n \left[\frac{t^{Nx}}{t^N - 1} - a\frac{t^{aNx}}{t^{Na} - 1}\right]\Bigg|_{t=1} \quad \text{and}\end{aligned}$$

$$L(1-n,\chi) = -\sum_{b=0}^{N} \chi(b)(1-a^{n+1})^{-1}\left(t\frac{d}{dt}\right)^n \left[\frac{t^b}{t^N - 1} - a\frac{t^{ab}}{t^{Na} - 1}\right]\Bigg|_{t=1}.$$

§1.13. Integrality.

Since $t^m - 1 = (t-1)(1 + t + \cdots + t^{m-1})$, we have

$$\Phi(t) := \frac{t^b}{t^N - 1} - a\frac{t^{ab}}{t^{Na} - 1} = \frac{t^b(1 + t^N + \cdots + t^{N(a-1)}) - at^{ab}}{(t-1)(\sum_{j=0}^{Na-1} t^j)}.$$

The numerator $\phi(t) = t^b(1 + t^N + \cdots + t^{N(a-1)}) - at^{ab}$ is divisible by $(t-1)$ as $\phi(1) = 0$, we do not have $t-1$ in the denominator of $\left(t\frac{d}{dt}\right)^n \Phi(t)$. Therefore $\Phi(t) \in \mathbb{Z}\left[\frac{1}{aN}\right][t]$.

Exercise: What is the GCD of $a^{n+1} - 1$ for all a prime to N? By this, give an estimate of possible denominators of $L(1-n, \chi)$.

We will see later that $L(1-n, \chi) \neq 0$ if $\chi(-1) = (-1)^n$, and therefore, $\phi(t)$ is not divisible by $(t-1)$ twice.

§**1.14. Euler products.** Consider a formal Dirichlet series $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, and suppose
(EP) $a_{mn} = a_m a_n$ *as long as* m *and* n *are co-prime.*

By prime factorization, if $m = \prod_p p^{e(n)}$, then $a_m = \prod_p a_{p^{e(n)}}$. Therefore, we get a formal expansion

$$L(s) = \prod_p \sum_{n=0}^{\infty} a_{p^n} p^{-ns}.$$

Consider the generating function for each p:

$$G_p(T) = \sum_{n=0}^{\infty} a_{p^n} T^n.$$

If $G_p(T)$ is a rational function $1/L_p(T)$ with $L_p \in \mathbb{C}[T]$ such that $D_p(0) = 1$ (reciprocally monic). We say that $L(s)$ has formal Euler product $L(s) = \prod_p L_p(p^{-s})^{-1}$, and when the product absolutely converges if Re$(s) > a$, we say $L(s)$ has Euler product (absolute convergence means $|L(s)| = \lim_{x\to\infty} \prod_{p<x} |L_p(p^{-s})|^{-1}$).

§**1.15. Euler product of Dirichlet L-function.** Plainly $a_n = \chi(n)$ is multiplicative. Note that $\sum_{n=0}^{\infty} \chi(p)^n T^n = \frac{1}{1-\chi(p)T}$ as long as $p \nmid N$. Thus we have

$$L(s,\chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}.$$

Exercise: Prove that the Euler product converges if Re$(s) > 1$.

By the convergence of Euler product means $|L_p(p^{-s})|^{-1} \to 1$ as $p \to \infty$. Since $L_p(p^{-s}) \neq 0$ for any p if Re$(s) > 1$, this implies if Re$(s) > 1$, we have $L(s,\chi) \neq 0$; so, by functional equation, $L(1-n,\chi) \neq 0$ if $\chi(-1) = (-1)^n$. If $n = 1$, one can show that $\prod_{j=0}^{h-1} L(s,\chi)$ has pole at $s = 1$. Since $\zeta(s)$ has a pole at $s = 1$, we know $L(0,\chi) \neq 0$ if $\chi(-1) = -1$.

§**1.16. Imprimitive Dirichlet L-function.** If $\chi(n) = \chi_0(n)$ for a primitive character modulo D for a divisor $D|N$, then

$$L(s,\chi) = \prod_{p\nmid N} (1-\chi(p)p^{-s})^{-1} = L(s,\chi_0) \prod_{p|N/D} (1-\chi(p)p^{-s}).$$

So analytic continuation of imprimitive L-functions follows from the result for primitive L-function.

In this lecture, we study Hecke L functions which is associated to Galois character χ sending the Frobenius element Frob_p at $p \nmid N$ to $\chi(p)$; i.e, it has values in $\mathrm{GL}_1(\mathbb{Q}(\chi))$. Galois representations π having values in $\mathrm{GL}_n(\mathbb{C})$ is called Artin L function. Let $L_p(\pi)(T) = \det(1_n - \pi(\mathrm{Frob}_p)T)$ (the reciprocal characteristic polynomial). Then we can make an (imprimitive) Artin L-function

$$L(s,\pi) = \prod_p L_p(\pi)(p^{-s})^{-1}.$$

There are many other interesting Euler products of degree n in Number theory.