We extend Hurwitz’s method of proving analytic continuation and functional equation to Dirichlet L-function. I suggest you to try working out Riemann’s method yourself. Take an integer \( N \geq 1 \). For a character \( \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \), Dirichlet L-function with character \( \chi \) is defined by \( L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s} \) which is absolutely convergent in the region \( \text{Re}(s) > 1 \). Here if \( n \) has a non-trivial common factor with \( N \), we agree to set \( \chi(n) = 0 \) as a convention. The residue ring \( \mathbb{Z}/N\mathbb{Z} \) surjects down to \( \mathbb{Z}/D\mathbb{Z} \) for any divisor \( D \) of \( N \); so, any character \( \chi_0 : (\mathbb{Z}/D\mathbb{Z})^\times \to \mathbb{C}^\times \) can be pulled back to a character of \( (\mathbb{Z}/N\mathbb{Z})^\times \). If there is no divisor \( D \) of \( N \) such that \( \chi \) comes from \((\mathbb{Z}/D\mathbb{Z})^\times\) in this way, \( \chi \) is called primitive.
§1.1. Hurwitz zeta function.

For $0 < x \leq 1$, we define

$$\zeta(s, x) := \sum_{n=0}^{\infty} (n + x)^{-s},$$

which is absolutely and locally uniformly convergent if $\text{Re}(s) > 1$. Indeed, if $\sigma = \text{Re}(s) > 1$, we have

$$|\zeta(s, x)| \leq \sum_{n=1}^{\infty} n^{-\sigma} \leq 1 + \int_{1}^{\infty} x^{-\sigma} \, dx = 1 + \left[ \frac{x^{-\sigma+1}}{-\sigma+1} \right]_{1}^{\infty} = 1 + (\sigma-1)^{-1}.$$

Note that

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \sum_{a=0}^{N} \chi(a)(Nn+a)^{-s} = \sum_{a=0}^{N} \chi(a)N^{-s}\zeta(s, \frac{a}{N}),$$

which is also absolutely and locally uniformly convergent if $\text{Re}(s) > 1$. Thus $\zeta(s, x)$ and $L(s, \chi)$ are holomorphic function in the right half plane $\{s \in \mathbb{C} | \text{Re}(s) > 1\}$. 
§1.2. Contour integral. Define $G(t, x) := \frac{e^{(1-x)t}}{e^t - 1} = \frac{e^{-tx}}{1 - e^{-t}} \quad (\ast)$

\[
\sum_{n=0}^{\infty} e^{-(n+x)t}.
\]

The identity $\ast$ is valid if $\text{Re}(t) > 0$. Note that $G(t, x) = O(e^{-tx})$ as $t \to \infty$ and $G(t, x) = O(t^{-1})$ as $t \to 0$ as the Laurent expansion of $G(t, x)$ starts with $t^{-1}$. Hence the Mellin transform $\int_0^\infty t^{s-1} G(t, x) dt$ converges if $\text{Re}(s) > 1$ and we get

\[
\int_0^\infty t^{s-1} G(t, x) dt = \Gamma(s) \zeta(s, x).
\]

Therefore allowing variable $z = re^{i\theta} \quad 0 \leq \theta \leq 2\pi \in \mathbb{C}$

\[
\int_{P(\varepsilon)} z^{s-1} G(z, x) dz = -\int_\varepsilon^\infty t^{s-1} G(t, x) dt + e^{2\pi i s} \int_\varepsilon^\infty t^{s-1} G(t, x) dt + \oint_{|z|=\varepsilon} z^{s-1} G(z, x) dz
\]

has limit as $\varepsilon \to 0$ when $\text{Re}(s) > 1$ given by

\[
\lim_{\varepsilon \to 0} \int_{P(\varepsilon)} z^{s-1} G(z, x) dz = (e^{2\pi i s} - 1) \Gamma(s) \zeta(s, x) \quad \text{if} \quad \text{Re}(s) > 1.
\]
§1.3. Meromorphic continuation.
By residue theorem, for $0 < \varepsilon' < \varepsilon < 1$, we have for the annulus $A(\varepsilon', \varepsilon)$ with inner radius $\varepsilon'$ and outer radius $\varepsilon$

\[
\int_{P(\varepsilon)} z^{s-1}G(z, x)\,dz - \int_{P(\varepsilon')} z^{s-1}G(z, x)\,dz = \int_{A(\varepsilon', \varepsilon)} G(z, x)\,dz = 0,
\]

and hence $s \mapsto \int_{P(\varepsilon)} z^{s-1}G(z, x)\,dz$ is well defined for all $s \in \mathbb{C}$ giving the identity $(e^{2\pi is} - 1)\Gamma(s)\zeta(s, x) = \int_{P(\varepsilon)} z^{s-1}G(z, x)\,dz$ which is an entire function of $s \in \mathbb{C}$. Therefore

\[
\zeta(s, x) = \frac{\int_{P(\varepsilon)} z^{s-1}G(z, x)\,dz}{(e^{2\pi is} - 1)\Gamma(s)}
\]

is a meromorphic function on $\mathbb{C}$ with possible poles at integers $n \leq 1$. Since $\Gamma(s)^{-1} = \frac{(s+n-1)(s+n-2)\cdots s}{\Gamma(s+n)}$, $\Gamma(s)^{-1}$ has simple zero at $-n$ ($0 \geq n \in \mathbb{Z}$). Thus the poles of $\zeta(s, x)$ is limited to $s = 1$.

Similarly we get $\Gamma(s) = \frac{\int_{P(\varepsilon)} z^{s-1}e^{-z}\,dz}{(e^{2\pi is} - 1)}$; so, $\Gamma(s)$ has simple pole at $s = 1 - n$ ($0 < n \in \mathbb{Z}$) with residue $\frac{(-1)^{n-1}(2\pi i)}{(n-1)!}$.

Exercise: Prove the above residue formula for $\Gamma(s)$. 
§1.4. Functional equation of $\zeta(s, x)$.

The function $G(z, x)$ has simple pole at $2\pi in$ for integers $n$ and

$$\text{Res}_{z=2\pi in} G(z, x) z^{s-1} = \begin{cases} e^{2\pi nx+(s-1)\pi i/2} |2n\pi|^{s-1} & \text{if } n > 0, \\ e^{-2\pi nx+3(s-1)\pi i/2} |2n\pi|^{s-1} & \text{if } n < 0. \end{cases}$$

This shows

$$\int_{D(m)} G(z, x) z^{s-1} \, dz = \frac{1}{2\pi i} \frac{1}{\Gamma(s)} \int_{P(\varepsilon)} G(z, x) z^{s-1} \, dz = \lim_{m \to \infty} \int_{D(m)} G(z, x) z^{s-1} \, dz$$

Assuming $\text{Re}(s) < 0$ and passing to the limit, we get

$$(e^{2\pi is} - 1) \Gamma(s) \zeta(s, x) = \int_{P(\varepsilon)} G(z, x) z^{s-1} \, dz = \lim_{m \to \infty} \int_{D(m)} G(z, x) z^{s-1} \, dz$$

$$= (2\pi)^s \left( e^{3\pi is/2} \sum_{n=1}^{\infty} e^{2\pi inx} n^{s-1} - e^{\pi is/2} \sum_{n=1}^{\infty} e^{-2\pi inx} n^{s-1} \right).$$
§1.5. Towards the functional equation of $L(s, \chi)$.

Plug in the functional equation of $\zeta(s, x)$ into

$$L(s, \chi) = \sum_{a=0}^{N} \chi(a) N^{-s} \zeta(s, \frac{a}{N}),$$

we get

$$(e^{2\pi is} - 1) \Gamma(s) L(s, \chi) =$$

$$\left(\frac{2\pi}{N}\right)^s (e^{3\pi is/2} \sum_{n=1}^{\infty} \sum_{a=0}^{N} \chi(a) e\left(\frac{na}{N}\right)n^{s-1} - e^{\pi is/2} \sum_{n=1}^{\infty} \sum_{a=1}^{N-1} \chi(a) e\left(-\frac{na}{N}\right)n^{s-1}).$$

Let $G(\chi) = \sum_{a=1}^{N-1} \chi(a) e\left(\frac{a}{N}\right)$. Then by $na \mapsto a$ (if $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, we get $\sum_{a=1}^{N-1} \chi(a) e\left(\frac{na}{N}\right) = \chi(n)^{-1} G(\chi)$. Assuming $\chi$ primitive, we will prove in the next page

$$\sum_{a=1}^{N-1} \chi(a) e\left(\frac{na}{N}\right) = 0 = \chi^{-1}(n) G(\chi) \text{ if } (n, N) := \text{GCD of } n \text{ and } N \neq 1.$$
§1.6. Gauss sum for primitive $\chi$.

**Lemma 1.** Let $G$ be a finite abelian group and $\chi : G \to \mathbb{C}^\times$ be a character. If $\chi$ is non-trivial, $\sum_{g \in G} \chi(g) = 0$.

Let $h$ be the order of $\chi$. Since the image $H$ of $\chi$ is a group $\mu_h$ of $h$-th roots of unity (which is cyclic), we may assume that $G$ is cyclic and $G \cong H$ by $\chi$. Then we have $\sum_{\zeta \in \mu_h} \zeta = \sum_{g \in G} \chi(g)$ which vanishes as it is the trace term of $X^h - 1$.

Suppose to have a prime $p|n$ with $n = pn'$ and $p|N$ with $N = pD$, and put $\psi(a) = e(a/N)$. Then

$$\sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)e(na/N) = \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)e(na/N)$$

$$= \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)e(n'a/D) = \sum_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} e(n'a/D)\chi(a) \sum_{b \equiv 1 \mod D} \chi(b),$$

which vanishes as $\chi$ restricted to $\{b \in (\mathbb{Z}/N\mathbb{Z})^\times | b \equiv 1 \mod D\}$ is nontrivial by primitivity.
§1.7. Functional equation of $L(s, \chi)$ of Euler type. By the discussion in §1.6, as long as $\chi$ is primitive, we have
\[
\sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)e(na/N) = \chi^{-1}(n)G(\chi)
\]
including $n$ not co prime to $N$ (following the convention $\chi^{-1}(n) = 0$ for such $n$). Therefore we obtain, if $\text{Re}(s) < 0$,
\[
(e^{2\pi is} - 1)\Gamma(s)L(s, \chi) = \left(\frac{2\pi}{N}\right)^s G(\chi)(e^{3\pi is/2} - \chi(-1)e^{\pi is/2})L(1-s, \chi).
\]
Since we already know the analytic continuation of $L(s, \chi)$, this identity is valid for all $s \in \mathbb{C}$. Since
\[
\frac{(e^{2\pi is} - 1)}{(e^{3\pi is/2} - \chi(-1)e^{\pi is/2})} = \begin{cases} 2\cos(\pi s/2) & \text{if } \chi(-1) = 1, \\ 2\sqrt{-1}\sin(\pi s/2) & \text{if } \chi(-1) = -1, \end{cases}
\]
we get
\[
L(s, \chi) = \begin{cases} G(\chi)(2\pi/N)^s L(1-s, \chi^{-1}) & \text{if } \chi(-1) = 1, \\ 2\Gamma(s)\cos(\pi s/2) & \text{if } \chi(-1) = -1, \\ G(\chi)(2\pi/N)^s L(1-s, \chi^{-1}) & \text{if } \chi(-1) = 1, \\ 2\sqrt{-1}\Gamma(s)\sin(\pi s/2) & \text{if } \chi(-1) = -1. \end{cases}
\]
§1.8. Functional equation of \( L(s, \chi) \) of Riemann type.

Put

\[
\hat{L}(s, \chi) = \begin{cases} 
\pi^{-s/2} \Gamma\left(\frac{s}{2}\right)L(s, \chi) & \text{if } \chi(-1) = 1, \\
\pi^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right)L(s, \chi) & \text{if } \chi(-1) = -1.
\end{cases}
\]

By \( \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \) (Euler) and

\[
\frac{\pi^{s-(1/2)}\Gamma((1-s)/2)}{\Gamma(s/2)} = \frac{(2\pi)^s}{2\Gamma(s)\cos(\pi s/2)}
\]

we have seen in §0.14 (or by Riemann’s method), we can interpret the functional equation in §1.7 into the following form:

\[
\hat{L}(s, \chi) = \kappa(\chi)N^{\frac{1}{2}-s}\hat{L}(1-s, \chi^{-1})
\]

with \( \kappa(\chi) = \begin{cases} G(\chi)/\sqrt{N} & \text{if } \chi(-1) = 1, \\
-\sqrt{-1}G(\chi)/\sqrt{N} & \text{if } \chi(-1) = -1.
\end{cases} \)

Thus \( \hat{L}(s, \chi) = \kappa(\chi)\kappa(\chi^{-1})\hat{L}(s, \chi) \) and hence \( |\kappa(\chi)| = 1 \).

Exercise: Why \( |\kappa(\chi)| = 1 \)?
1.9. Bernoulli polynomials.

Let $F(z, x) := zG(z, 1-x) = \frac{ze^{zx}}{e^z - 1}$ and expand $F(z, x)$ into a power series in $z$: $F(z, x) = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}$. Since $e^{zx} = \sum_{n=0}^{\infty} \frac{x^n z^n}{n!}$, $B_n(x)$ is a polynomial of $x$. Note that $F(z, x) = e^{zx} \frac{ze^z - 1}{e^z - 1} = \left( \sum_{n=0}^{\infty} x^n \frac{z^n}{n!} \right) \left( \sum_{m=0}^{\infty} B_m \frac{z^m}{m!} \right)$. Thus we get

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j} \quad \text{and}$$

$$\text{Res}_{z=0} G(z, x) z^{-n} = \text{Res}_{z=0} F(z, 1-x) z^{-n-1} = \frac{B_n(1-x)}{n!}.$$

Since $F(z, 1-x) = \frac{ze^{(1-x)z}}{e^z - 1} = \frac{ze^{-xz}}{1-e^{-z}} = F(-z, x)$, we get

$$B_n(1-x) = (-1)^n B_n(x).$$

Examples: $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{x}{6}$, ...
§1.10. The value of $\zeta(s, x)$ at $s = 1 - n \leq 0$. Let $\mathbb{Q}(\chi)$ be the field inside $\mathbb{C}$ generated by the values of $\chi$. Thus writing $h$ for the order of $\chi$, $\mathbb{Q}(\chi) = \mathbb{Q}(\mu_h)$ is the field of $h$-th roots of unity. By the formula given in §1.3, we have

$$(e^{2\pi is} - 1)\Gamma(s)\zeta(s, x) = \int_{P(\varepsilon)} z^{s-1}G(z, x)dz.$$ 

If $s$ is an integer $1 - n$ ($0 < n \in \mathbb{Z}$), then $z^{s-1}G(z, x)$ is single valued, and by $(e^{2\pi is} - 1)\Gamma(s)|_{s=1-n} = \frac{(-1)^{n-1}2\pi i}{(n-1)!}$, the formula becomes

$$\frac{(-1)^{n-1}2\pi i}{(n-1)!}\zeta(1 - n, x) = \int_{\partial D_\varepsilon} z^{-n}G(z, x)dz$$

$$= 2\pi i \text{Res}_{z=0}z^{-n}G(z, x) = 2\pi i \frac{B_n(1 - x)}{n!} = (-1)^n2\pi i \frac{B_n(x)}{n!}$$

for the circle of radius $\varepsilon$ centered at the origin.
1.11. Rationality Theorem. We have, for $0 < n \in \mathbb{Z}$

$$\zeta(1 - n, x) = \frac{B_n(x)}{n} \quad \text{and}$$

$$L(1 - n, \chi) = - \sum_{b=0}^{N} \chi(b) N^n \frac{B_n(b/N)}{n} \in \mathbb{Q}(\chi).$$

If $\chi(-1) \neq (-1)^n$ and $\chi \neq 1$, $L(1 - n, \chi) = 0$, and $\zeta(1 - n) = 0$ if $n > 1$ is odd. In the above formula, $\chi$ can be any function on $\mathbb{Z}/N\mathbb{Z}$.

The last assertion follows from

$$L(1 - n, \chi) \overset{b \to N-b}{=} - \sum_{b=0}^{N} \chi(-b) N^n \frac{B_n((N-b)/N)}{n}$$

$$= -(-1)^n \chi(-1) \sum_{b=0}^{N} \chi(b) N^n \frac{B_n(b/N)}{n} = (-1)^n \chi(-1)L(1 - n, \chi)$$

by $B_n((N - b)/N) = (-1)^n B_n(b/N)$. 

§1.12. Rational function expression. Put \( t = e^{z/N} \). Then

\[
\frac{t^N x}{t^N - 1} = \frac{1}{z} F(z, x) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{B_{n+1}(x)}{(n + 1)!} z^n
\]

and for any integer \( a > 1 \),

\[
\frac{t^N x}{t^N - 1} - a \frac{t^a N x}{t^{Na} - 1} = \sum_{n=0}^{\infty} \frac{B_{n+1}(x) - a^{n+1} B_{n+1}(x)}{(n + 1)!} z^n
\]

and by \( dt^N = N t^{N-1} dt \Leftrightarrow N t^N \frac{d}{dt^N} = \frac{d}{dt} \)

\[
(1 - a^{n+1}) N^n \frac{B_{n+1}(x)}{n + 1} = N^n \left( t^N \frac{d}{dt^N} \right)^n \left[ \frac{t^N x}{t^N - 1} - a \frac{t^a N x}{t^{Na} - 1} \right] \bigg|_{t=1}
\]

\[= \left( \frac{d}{dt} \right)^n \left[ \frac{t^N x}{t^N - 1} - a \frac{t^a N x}{t^{Na} - 1} \right] \bigg|_{t=1} \text{ and}
\]

\[L(1-n, \chi) = - \sum_{b=0}^{N} \chi(b)(1-a^{n+1})^{-1} \left( \frac{d}{dt} \right)^n \left[ \frac{t^b}{t^N - 1} - a \frac{t^{ab}}{t^{Na} - 1} \right] \bigg|_{t=1}.
\]
§1.13. Integrality.
Since \( t^m - 1 = (t - 1)(1 + t + \cdots + t^{m-1}) \), we have
\[
\Phi(t) := \frac{t^b}{t^N - 1} - a\frac{t^{ab}}{t^{Na} - 1} = \frac{t^b(1 + t^N + \cdots + t^{N(a-1)}) - at^{ab}}{(t - 1)(\sum_{j=0}^{Na-1} t^j)}.
\]

The numerator \( \phi(t) = t^b(1 + t^N + \cdots + t^{N(a-1)}) - at^{ab} \) is divisible by \( (t - 1) \) as \( \phi(1) = 0 \), we do not have \( t - 1 \) in the denominator of \( \left( \frac{t^d}{dt} \right)^n \Phi(t) \). Therefore \( \Phi(t) \in \mathbb{Z} \left[ \frac{1}{aN} \right] [t] \).

Exercise: What is the GCD of \( a^{n+1} - 1 \) for all \( a \) prime to \( N \)? By this, give an estimate of possible denominators of \( L(1 - n, \chi) \).

We will see later that \( L(1 - n, \chi) \neq 0 \) if \( \chi(-1) = (-1)^n \), and therefore, \( \phi(t) \) is not divisible by \( (t - 1) \) twice.
§1.14. **Euler products.** Consider a formal Dirichlet series
\[ L(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \]
and suppose
\[ (EP) \ a_{mn} = a_m a_n \text{ as long as } m \text{ and } n \text{ are co-prime.} \]

By prime factorization, if \( m = \prod_p p^{e(n)} \), then \( a_m = \prod_p a_{p^{e(n)}} \).

Therefore, we get a formal expansion
\[ L(s) = \prod_p \sum_{n=0}^{\infty} a_p n p^{-ns}. \]

Consider the generating function for each \( p \):
\[ G_p(T) = \sum_{n=0}^{\infty} a_p n T^n. \]

If \( G_p(T) \) is a rational function \( 1/L_p(T) \) with \( L_p \in \mathbb{C}[T] \) such that \( D_p(0) = 1 \) (reciprocally monic). We say that \( L(s) \) has formal Euler product \( L(s) = \prod_p L_p(p^{-s})^{-1} \), and when the product absolutely converges if \( \text{Re}(s) > a \), we say \( L(s) \) has Euler product (absolute convergence means \( |L(s)| = \lim_{x \to \infty} \prod_{p<x} |L_p(p^{-s})|^{-1} \)).
§1.15. Euler product of Dirichlet L-function. Plainly $a_n = \chi(n)$ is multiplicative. Note that $\sum_{n=0}^{\infty} \chi(p^n)T^n = \frac{1}{1-\chi(p)T}$ as long as $p \nmid N$. Thus we have

$$L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}.$$ 

Exercise: Prove that the Euler product converges if $\text{Re}(s) > 1$.

By the convergence of Euler product means $|L_p(p^{-s})|^{-1} \to 1$ as $p \to \infty$. Since $L_p(p^{-s}) \neq 0$ for any $p$ if $\text{Re}(s) > 1$, this implies if $\text{Re}(s) > 1$, we have $L(s, \chi) \neq 0$; so, by functional equation, $L(1-n, \chi) \neq 0$ if $\chi(-1) = (-1)^n$. If $n = 1$, one can show that $\prod_{j=0}^{h-1} L(s, \chi)$ has pole at $s = 1$. Since $\zeta(s)$ has a pole at $s = 1$, we know $L(0, \chi) \neq 0$ if $\chi(-1) = -1$. 
§1.16. **Imprimitive Dirichlet L-function.** If \( \chi(n) = \chi_0(n) \) for a primitive character modulo \( D \) for a divisor \( D|N \), then

\[
L(s, \chi) = \prod_{p \nmid N} (1 - \chi(p)p^{-s})^{-1} = L(s, \chi_0) \prod_{p \mid N/D} (1 - \chi(p)p^{-s}).
\]

So analytic continuation of imprimitive L-functions follows from the result for primitive L-function.

In this lecture, we study Hecke L functions which is associated to Galois character \( \chi \) sending the Frobenius element \( \text{Frob}_p \) at \( p \nmid N \) to \( \chi(p) \); i.e, it has values in \( \text{GL}_1(\mathbb{Q}(\chi)) \). Galois representations \( \pi \) having values in \( \text{GL}_n(\mathbb{C}) \) is called Artin L function. Let \( L_p(\pi)(T) = \det(1_n - \pi(\text{Frob}_p)T) \) (the reciprocal characteristic polynomial). Then we can make an (imprimitive) Artin L-function

\[
L(s, \pi) = \prod_p L_p(\pi)(p^{-s})^{-1}.
\]

There are many other interesting Euler products of degree \( n \) in Number theory.