* Hecke L-functions and their critical values

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0.0. Introduction We plan to discuss the following four topics:

- Analytic continuation/functional equation of Dirichlet L,
- Analytic continuation/functional equation of Hecke L functions for totally real field (and possibly for general number fields),
- Rationality and integrality of L-values (via Shintani),
- If time allows, construction of Kubota-Leopoldt and Deligne-Ribet p-adic L functions.

In this introduction, we describe Euler's way (1749) of computing $\zeta(-n)$ for $0 \leq n \in \mathbb{Z}$ without having analytic continuation. Euler justified his computation relating $\zeta(1-2n)$ with $\zeta(2n)$ without having functional equation (or having functional equation valid only for integers), as $\zeta(2n)$ converges absolutely. Then in the following lectures, we justifies giving an analytic continuation via the method of Riemann (1859) and Hurwitz (1882), which we generalize by a method of Shintani (1976). These explicit formulas allow very elementary construction of p-adic L functions (Cassou-Nogués 1979, Deligne-Ribet 1980, Katz 1981).

§0.1. Zeta values at non-positive integers

Consider the altrnating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}=\sum_{n=1}^{\infty}n^{-s}-2\sum_{n=1}^{\infty}(2n)^{-s}=(1-2^{1-s})\zeta(s).$$

Replacing n^{-s} by t^n, we get a geometric series

$$g(t)=\sum_{n=1}^{\infty}(-1)^{n-1}t^n=-\sum_{n=1}^{\infty}(-t)^n=\frac{t}{1+t}.$$

Thus for $\partial:=t\frac{d}{dt}$, $\partial t^n=nt^n$, and we have

$$\partial^m g(t)=\sum_{n=1}^{\infty}(-1)^{n-1}n^m t^n.$$

Since $t-1$ is not a denominator of $\partial^n g(t)$, we would have

$$(1-2^{1+m})\zeta(-m)=\partial^m g(t)|_{t=1}\in\mathbb{Q}.$$

§0.2. Trigonometric functions

Recall from Euler's formula $z = r(\cos\theta + i\sin\theta)$, we have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \ \sin(z) = \frac{e^{iz} - e^{-iz}}{2\sqrt{-1}},$$

$$\cot(z) = \sqrt{-1}\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = \sqrt{-1}\frac{e^{2iz} + 1}{e^{2iz} - 1}.$$

and its partial fraction expansion [LFE, p. 30–31] or [EDM, 2.4]:

$$\varepsilon_1(z) := \pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left\{\frac{1}{z+n} + \frac{1}{z-n}\right\}.$$

From $\frac{1}{z \pm n} = \pm n^{-1}\sum\frac{1}{1 \pm n^{-1}z} = \pm\sum_{r=0}(\mp 1)^r n^{-r-1}z^r$, we get

$$\varepsilon_1(z) = \frac{1}{z} - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} n^{-2k}z^{2k-1} = \frac{1}{z} - 2\sum_{k=1}^{\infty}\zeta(2k)z^{2k-1}.$$

By the way, Weierstrass $\wp$ and ζ function is a direct generalization of the above expansion of ε_1 to the theory of elliptic curves.

§**0.3. Euler's justification.** By putting $t = e^x$, ∂ corresponds to the translation invariant differential operator $\frac{d}{dx}$, hence ∂ is the invariant differential operator on $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$. Write $f(x) = g(e^x)$. Then the formula becomes

$$(1 - 2^{m+1})\zeta(-m) = \left\{ \left(\frac{d}{dx}\right)^m f(x) \right\} |_{x=0}$$

Therefore, Taylor expansion of $F(z) = g(\mathbf{e}(z))$ with $\mathbf{e}(z) = \exp(2\pi i z)$, at $z = 0$ is given by

$$F(z) = \sum_{n=0}^{\infty} \frac{F^{(n)}}{n!} z^n = \sum_{n=0}^{\infty} \frac{(2\pi i)^n (1 - 2^{n+1})}{n!} \zeta(-n) z^n.$$

On the other hand,

$$\begin{aligned} F(-z) - F(z) &= \frac{\mathbf{e}(-z)}{1 + \mathbf{e}(-z)} - \frac{\mathbf{e}(z)}{1 + \mathbf{e}(z)} \\ &= \frac{\mathbf{e}(z) + 1}{\mathbf{e}(z) - 1} - 2\frac{\mathbf{e}(2z) + 1}{\mathbf{e}(2z) - 1} = \frac{\varepsilon_1(z) - 2\varepsilon_1(2z)}{\pi i} \end{aligned}$$

§**0.4. Functional equation at integer level.** We then have

$$-\sum_{k=1}^{\infty}(2\pi i)^{2k-1}2(1-2^{2k})\zeta(1-2k)\frac{z^{2k-1}}{(2k-1)!}=F(-z)-F(z)$$
$$=\frac{\varepsilon_1(z)-2\varepsilon_1(2z)}{\pi i}=\frac{-1}{\pi i}\sum_{k=1}^{\infty}2(1-2^{2k})\zeta(2k)z^{2k-1}.$$

Comparing coefficients in z^{2k-1}, we get

$$\zeta(2k)=\frac{(2\pi)^{2k}}{2(2k-1)!(-1)^k}\zeta(1-2k).$$

Euler conjectured that $\zeta(s)$ has meromorphic continuation to the whole $s\in\mathbb{C}$ with a simple pole at $s=1$ satisfying

$$\zeta(s)=\frac{(2\pi)^s\zeta(1-s)}{2\Gamma(s)\cos(\pi s/2)}\quad(\forall s\in\mathbb{C}).$$

Here $\Gamma(s)$ is Euler's gamma function given by $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$, which is convergent plainly if $\mathrm{Re}(s)>0$. Therefore we have

$$a^{-s}\Gamma(s)=\int_0^\infty t^{s-1}e^{-at}dt\quad(0<\forall a\in\mathbb{R}).$$

§0.5. Euler's Gamma function, 1729.

Since $\frac{de^{-t}}{dt} = -e^{-t}$ and $\frac{dt^s}{dt} = st^{s-1}$, we have $(-e^{-t}t^s)' = e^{-t}t^s - se^{-t}t^{s-1}$. Integrating this from 0 to ∞, we get

$$0 = \left[-e^{-t}t^s\right]_0^\infty = \Gamma(s+1) - s\Gamma(s).$$

We get

$$\Gamma(s+1) = s\Gamma(s) \quad \text{if } \mathrm{Re}(s) > 0.$$

Therefore $\Gamma(s)$ has meromorphic continuation with simple poles at non-positive integers, and $\Gamma(s)^{-1}$ is an entire function on the whole $\mathbb{C}$. Note that

$$\Gamma(1) = \int_0^\infty e^{-t}dt = \left[-e^{-t}\right]_0^\infty = 1.$$

This combined with $\Gamma(s+1) = s\Gamma(s)$ tells us $\Gamma(n) = (n-1)!$. We also have the following contour integral expression

$$\Gamma(s) = (\mathbf{e}(s) - 1)^{-1} \int_{P(\varepsilon)} e^{-z}z^{s-1}dz \quad (\forall s \in \mathbb{C}),$$

where $P(\varepsilon)$ is a contour from ∞ to $\varepsilon > 0$ going up over real axis on the circle of radius ε counter-clock wise and going back to ∞ from ε.

§**0.6. Fourier transform.** For an integrable function $f : \mathbb{R} \to \mathbb{C}$, we define its Fourier transform by $\widehat{f}(x) = \int_{-\infty}^{\infty} f(y)\mathbf{e}(-xy)dx$. Take $f(x) = \exp(-\pi x^2)$. By a formula in any Calculus book, we have $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \exp(-\pi x^2)dx = 1$. By the variable change $x \mapsto (\sqrt{a/\pi})x$ for $a > 0$, we get $\int_{-\infty}^{\infty} \exp(-ax^2)dx = \sqrt{\pi/a}$.

Lemma 1. *For $f(x) = \exp(-\pi x^2)$, we have $\widehat{f}(x) = f(x)$.*

Proof We claim $g(z) := \int_{-\infty}^{\infty} \exp(-\pi(x + z)^2)dx = 1$ for all $z = u + it \in \mathbb{C}$. By computation for $x \in \mathbb{R}$,

$$\exp(-\pi(x+u+it)^2) = \exp(-\pi(x+u)^2)\exp(-2\pi(x+u)it)\exp(\pi t^2).$$

This shows the integral is uniformly convergent in a compact set of $\mathbb{C}$, and hence differentiation by $\frac{d}{d\overline{z}}$ commutes with integration; i.e., $g(z)$ is holomorphic in z. A holomorphic function has convergent power series expansion determined by its restriction to $\mathbb{R}$; so, $g(z) = 1$. Taking $z = it$, we get the result. $\square$

§0.7. Poisson summation formula (around 1820).

Theorem 1. *If* $f : \mathbb{R} \to \mathbb{C}$ *is continuous and integrable, then*

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m)\mathbf{e}(mx)$$

provided the two sides converge absolutely and uniformly.

Sketch of Proof: Let $g(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ which is continuous and integer translation invariant. Hence it has Fourier expansion

$$g(x) = \sum_{m\in\mathbb{Z}} c_m \mathbf{e}(mx) \quad \text{with } c_m = \int_0^1 g(x)\mathbf{e}(-mx)dx \quad \text{and}$$

$$c_m = \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)\mathbf{e}(-m(x+n))dx = \int_{-\infty}^{\infty} f(x)\mathbf{e}(-mx)dx = \widehat{f}(m).$$

proves the identity. $\square$

One can prove the partial farction expansion of the cotangent function by Poisson summation [EDM, 2.4].

§0.8. Mellin transform of $\theta(t)$.

Poisson summation (evaluated at $x = 0$) applied to $f(x) = \exp(-\pi t x^2)$ with $\widehat{f}(x) = \sqrt{t}^{-1} \exp(\pi t^{-1} x^2)$ produces

$$\sum_{n\in\mathbb{Z}} \exp(-\pi t n^2) = \sqrt{t}^{-1} \sum_{m\in\mathbb{Z}} \widehat{f}(m) \exp(-\pi t n^2).$$

Thus

$$\theta(t) := \sum_{n\in\mathbb{Z}} \exp(-\pi t n^2) = \sqrt{t}^{-1} \sum_{m\in\mathbb{Z}} \exp(-\pi t^{-1} m^2) = \sqrt{t}^{-1} \theta(t^{-1}). \tag{PS}$$

Note $\Theta := (\theta(t) - 1)/2 = \sum_{n=1}^{\infty} e^{-\pi t n^2}$. If $\mathrm{Re}(s) > 1$, for $d\mu = dt/t$,

$$\begin{aligned}\int_0^1 t^{s/2}\theta(t)/2 d\mu - \frac{1}{2}\int_0^1 t^{s/2-1} d\mu + \int_1^\infty t^{s/2}\Theta d\mu &= \int_0^\infty t^{s/2}\Theta d\mu \\ &\overset{(*)}{=} \sum_{n=1}^{\infty} \int_0^\infty t^{s/2} e^{-\pi t n^2} d\mu = \pi^{-s}\Gamma(s/2)\zeta(s).\end{aligned}$$

The interchange of integral and summation at $(*)$ follows from Lebesgue's dominated convergence theorem as

$$\Theta(t) = O(e^{-\pi t}) \quad \text{as } t \to \infty, \text{ and } \quad \Theta(t) = O(t^{-1/2}) \quad \text{as } t \to 0.$$

§**0.9. Riemann's justification of Euler's formula, 1859.** By $-\frac{dt^{-1}}{t^{-1}} = \frac{dt}{t}$, for $\Theta := (\theta(t)-1)/2$ and $\Theta' := t^{-1/2-s}(\theta(t)-1)$,

$$
\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \int_0^1 t^{s/2}\theta(t)/2d\mu - \frac{1}{2}\int_0^1 t^{s/2-1}d\mu + \int_1^\infty t^{s/2}\Theta d\mu \\
&\overset{\text{(PS)}}{=} \int_0^1 t^{s/2-1/2}\theta(t^{-1})/2d\mu - \frac{1}{2}\left[\frac{t^{s/2}}{s/2}\right]_0^1 + \int_1^\infty t^{s/2}\Theta d\mu \\
&\overset{t\mapsto t^{-1}}{=} \int_1^\infty t^{1/2-s/2}(\theta(t)-1)/2d\mu + \frac{1}{2}\left[\frac{t^{1/2-s/2}}{1/2-s/2}\right]_1^\infty - \frac{1}{s} + \int_1^\infty t^{s/2}\Theta d\mu \\
&= \int_1^\infty t^{1/2-s/2}\Theta d\mu + \int_1^\infty t^{s/2}\Theta d\mu - \frac{1}{s} + \frac{1}{s-1} \quad \text{(S)}
\end{aligned}
$$

The equation (1) is invariant under $s \mapsto 1-s$; so,

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{(s-1)/2}\Gamma((1-s)/2)\zeta(1-s)$$

and $\widehat{\zeta}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ has simple at $s = 0$ and $s = 1$ satisfying $\widehat{\zeta}(s) = \widehat{\zeta}(1-s)$. The two integrals in (S) converge everywhere giving meromorphic continuation of $\zeta(s)$.

§0.10. Hurwitz justification, 1882.

Let $G(z) = \frac{1}{e^z - 1} = \frac{e^{-z}}{1 - e^{-z}} = \sum_{n=1}^{\infty} e^{-nz}$. Note that $G(t) = O(e^{-t})$ as $t \to \infty$ and $G(t) = O(t^{-1})$ as $t \to 0$. Thus if $\mathrm{Re}(s) > 1$, the Mellin transform $\int_0^\infty G(t) t^s d\mu$ converges if $\mathrm{Re}(s) > 1$ and the geometric series sum giving $G(t)$ and the integral can be interchanged to have

$$\Gamma(s)\zeta(s) = \int_0^\infty G(t) t^s d\mu = \int_0^\infty G(z) z^{s-1} dz.$$

If $\mathrm{Re}(s) > 1$, $G(z)z^{s-1}$ is bounded at the origin. Therefore

$$\lim_{\varepsilon \to 0} \int_{P(\varepsilon)} G(z) z^{s-1} dz = (e^{2\pi i s} - 1)\Gamma(s)\zeta(s) \quad \text{if } \mathrm{Re}(s) > 1.$$

Since $G(z)z^{s-1}$ does not have pole in $P(\varepsilon) - P(\varepsilon')$ $0 < \varepsilon' < \varepsilon < 1$, by the residue theorem,

$$\int_{P(\varepsilon)} G(z) z^{s-1} dz = \int_{P(\varepsilon')} G(z) z^{s-1} dz.$$

Thus we have $s \mapsto \int_{P(\varepsilon)} G(z) z^{s-1} dz$ independent of ε giving an entire function of all $s \in \mathbb{C}$.

§0.11. Meromorphic continuation. By the argument in §0.11,

$$\zeta(s) = \frac{\int_{P(\varepsilon)} G(z) z^{s-1} dz}{(e^{2\pi i s} - 1)\Gamma(s)}$$

gives meromorphic function over all $s \in \mathbb{C}$.

Exercise: The location of only simple poles at $s = 0, 1$ of $\widehat{\zeta}(s)$ by Riemann tells us what property of $\Gamma(s)$?

As meromorphic function of $z \in \mathbb{C}$, $G(z)$ has simple poles at $z = 2m\pi i$ for $m \in \mathbb{Z}$. Since $G(z + 2\pi i) = G(z)$, by the form $G(z) = \frac{1}{e^z - 1}$, $\text{Res}_{z=0} G(z) = 1$. We cut $\mathbb{C}$ by the positive real line. Writing $z = re^{i\theta}$ for $0 \le \theta \le 2\pi$ and defining $z^s = r^s e^{is\theta}$. This shows

$$\text{Res}_{z=2\pi i n}(G(z) z^{s-1}) = \begin{cases} -\sqrt{-1}|2n\pi|^{s-1} e^{s\pi i/2} & \text{if } n > 0, \\ \sqrt{-1}|2n\pi|^{s-1} e^{3s\pi i/2} & \text{if } n < 0. \end{cases}$$

§**0.12. Hurwitz's contour.** Let $D(m)$ be the union of $P(\varepsilon, m) := P(\varepsilon) \cap D_m$ for the square D_m centered at the origin of edge length $\pi(4m+2)$ for positive integer m and ∂D_m. The orientation of ∂D_m is clock-wise. Then we get from the residue theorem

$$\int_{D(m)} G(z) z^{s-1} dz = (2\pi)^s e^{\pi i s}(e^{\pi i s/2} - e^{-\pi i s/2}) \sum_{n=1}^{m} n^{s-1}.$$

Then we have an absolutely convergent limit: if $\mathrm{Re}(s) < 0$,

$(*) := \lim_{m\to\infty} \int_{D(m)} G(z) z^{s-1} dz = (2\pi)^s e^{\pi i s}(e^{\pi i s/2} - e^{-\pi i s/2})\zeta(1-s)$ and

$$\begin{aligned}(*) &= \lim_{m\to\infty} \int_{P(\varepsilon,m)} G(z) z^{s-1} dz + \lim_{m\to\infty} \int_{\partial D_m} G(z) z^{s-1} dz \\ &= \int_{P(\varepsilon)} G(z) z^{s-1} dz + \lim_{m\to\infty} \int_{\partial D_m} G(z) z^{s-1} dz \\ &= (e^{2\pi i s} - 1)\Gamma(s)\zeta(s) + \lim_{m\to\infty} \int_{\partial D_m} G(z) z^{s-1} dz.\end{aligned}$$

§**0.13. Vanishing of** $\lim_{m\to\infty}\int_{\partial D_m} G(z)z^{s-1}dz$**.** Write $\partial D_m^{\pm}$ (resp. $\partial D_{m,\pm}$) for upper and lower edges (left and right edges) of ∂D_m. We give an estimate of $|\lim_{m\to\infty}\int_{\partial D_m^+} G(z)z^{s-1}dz|$, and the other cases can be dealt with similarly. Write $z = x + \pi i(2m+1)$ with $x \in \mathbb{R}$. Then $e^{-z} = -e^{-x} > 0$ and $|1 - e^{-z}| = 1 + e^{-x} > e^{-x}$; so, $|G(z)z^{s-1}| \le C|(2m+1)\pi i + x|^{\sigma-1}$ ($\sigma = \mathrm{Re}(s)$) on ∂D_m^+ for $C > 0$ dependent only on σ, and for $M := 2m+1$

$$
\begin{aligned}
|\int_{\partial D_m^+} G(z)z^{s-1}dz| &= |\int_{-M}^{M} G(M\pi i + x)(M\pi i + x)^{s-1}dx| \\
&\le \int_{-M}^{M} |G(M\pi i + x)(M\pi i + x)^{s-1}|dx \\
&\le C\int_{-M}^{M} |M\pi i + x|^{\sigma-1}dx = C\int_{-M}^{M}(M^2\pi^2 + x^2)^{(\sigma-1)/2}dx \\
&< C\int_{-M}^{M}(M\pi)^{\sigma-1}dx = 2C\pi^{\sigma-1}M^{\sigma} \to 0 \ \text{ as } m\to\infty \text{ if } \sigma < 0.
\end{aligned}
$$

Exercise: prove $\lim_{m\to\infty}\int_{\partial D_{m,+}} G(z)z^{s-1}dz = 0$ if $\sigma < 0$.

§**0.14. Functional equation.** By computation in §0.12 and §0.13 combined, we have if Re(s) < 0,

$$(2\pi)^s e^{\pi is}(e^{\pi is/2} - e^{-\pi is/2})\zeta(1-s) = (e^{2\pi is} - 1)\Gamma(s)\zeta(s).$$

Multiplying by $e^{-\pi is}$, we get

$$(e^{\pi is} - e^{-\pi is})\Gamma(s)\zeta(s) = (2\pi)^s(e^{\pi is/2} - e^{-\pi is/2})\zeta(1-s).$$

Since $\frac{e^{\pi is} - e^{-\pi is}}{e^{\pi is/2} - e^{-\pi is/2}} = e^{\pi s/2} + e^{-\pi s/2} = 2\cos(\pi s/2)$, we get, as Euler predicted,

$$\zeta(s) = \frac{(2\pi)^s \zeta(1-s)}{2\Gamma(s)\cos(\pi s/2)}.$$

Comparing with Riemann's functional equation, we get

$$\frac{\pi^{s-(1/2)}\Gamma((1-s)/2)}{\Gamma(s/2)} = \frac{(2\pi)^s}{2\Gamma(s)\cos(\pi s/2)}.$$

Exercise: prove the above formula directly without using Riemann ζ-function.