Sets (1.1)

1. Determine the cardinality of (aka size of, aka number of elements in) the following sets.
   (a) \( A = \{1, 2, 3\} \)
   (b) \( B = \{\{1, 2, 3\}\} \)
   (c) \( C = \{\{1, 2, 3\}, 4\}, 5\} \)
   (d) \( D = \{x \in \mathbb{Z} : |3x| \leq 10\} \)

2. Explicitly write out all elements of the following sets.
   (a) \( A = \{x \in \mathbb{Z} : x \in [-\pi, \pi]\} \)
   (b) \( B = \{x \in \mathbb{Z} : |2x + 1| \leq 9\} \)
   (c) \( C = \{2x + 1 : x \in B\} \)
   (d) \( D = \{x \in \mathbb{R} : x^2 = 3\} \)
   (e) \( E = \{x \in \mathbb{Z} : x^2 = 3\} \)
   (f) \( F = \{x \in \mathbb{N} : x^2 = 9\} \)
   (g) \( G = \{x \in \mathbb{Z} : x^2 \leq 3\} \)
   (h) \( H = \{x \in B : \cos(\pi x) = 1\} \)

3. Try to write out the elements of the following sets. You may need to use ... if there are infinitely many elements.
   (a) \( \mathbb{Z}_{\geq 0} = \{x \in \mathbb{Z} : x \geq 0\} \)
   (b) \( A = \{2a + 3b : a, b \in \mathbb{Z}_{\geq 0}\} \) (Bonus: Can you interpret this set in terms of basketball?)
   (c) \( B = \{a + 2b : a \in \{0, 1\}, b \in \mathbb{Z}\} \)
   (d) \( 2B = \{2x : x \in B\} \)
   (e) \( C = \{5x : x \in \mathbb{N}, |2x| > 8\} \)
   (f) \( D = \{x \in \mathbb{R} : \sin\left(\frac{\pi x}{2}\right) = 1\} \)
   (g) \( E = \{x \in \mathbb{Z} : x^2 - 5x + 6 > 0\} \)
   (h) \( F = \{x \in \mathbb{N} : x^2 - 5x + 6 > 0\} \)

4. Find a way to use set-builder notation to write the following sets more succinctly.
   (a) \( \{6, 9, 12, 15, 18, \ldots, 108\} \)
   (b) \( \{1, 9, 25, 49, 81, \ldots\} \)
   (c) \( \{\ldots, \frac{7}{2}, \frac{5}{2}, 2, 6, 18, 54, \ldots\} \)
   (d) \( \{(1, 1), (2, 2), (3, 3), (4, 4), \ldots\} \)
   (e) \( \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\} \)
   (f) \( \{1, 2, \ldots, 1024\} \)
   (Can you come up with two different answers to this question? This is why we avoid ...)

5. Draw the following sets on the number line or the \(xy\)-plane, as appropriate.
   (a) \( \{x \in \mathbb{R} : x^2 - 5x + 6 \geq 0, x > 2\} \)
   (b) \( \{(x, y) \in \mathbb{R}^2 : x^2 < y < x\} \)
   (c) \( \{(x, y) \in \mathbb{R}^2 : y = 5x - 1, x \in (-1, 1)\} \)
   (d) \( \{(x, y) \in \mathbb{R}^2 : y = 5x - 1, x \in (-1, 1)\} \)
Cartesian Products (1.2)

1. Explicitly write out all elements in the following sets. What is the cardinality of the set?
   (a) \( \{1, 2, 3, 4\} \times \{1, 2\} \)
   (b) \( \{1, 2, 3, 4\} \times \{(1, 2)\} \)
   (c) \( \{1, 2, \{3, 4\}\} \times \{1, \{2\}\} \)
   (d) \( \{1, 2\} \times \{a, b\} \times \{x, y\} \)
   (e) \( \{\pi, e\}^2 \)
   (f) \( \{\pi, e\} \times \emptyset \)
   (g) \( \{\pi, e\} \times \{\emptyset\} \)
   (h) \( \{x \in \mathbb{R} : x^2 = 9\} \times \{x \in \mathbb{R} : |x| = 9\} \)

2. Draw the following sets on the \( xy \)-plane, or in 3D space, as appropriate.
   (a) \( A^2 \), where \( A = \{0, 1\} \)
   (b) \( \{0, 1\} \times [0, 1] \)
   (c) \( [0, 1] \times [0, 1] \)
   (d) \( [0, 1] \times [0, 2] \times [0, 3] \)
   (e) \( \mathbb{N} \times \mathbb{N} \)
   (f) \( [1, 2] \times \{1, 2, 3\} \)
   (g) \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \times [0, 1] \)
   (h) \( [1, 0] \times [0, 1] \times \{0, 0\} \)

3. Write the following sets more succinctly using Cartesian products.
   (a) \( \{(x, y) : x \in \mathbb{R}, y \in \mathbb{Z}, y > 0\} \)
   (b) \( \{(x, y), z) : x \in \mathbb{R}, y \in \mathbb{Z}, z \in \{0, 1\}\} \)
   (c) \( \{(0, 0), (1, 0), (1, 1), (0, 2), (0, 1), (1, 2), (0, 1), (0, 0)\} \)
   (d) The set of vertices of the unit cube \([0, 1]^3\)

4. If possible, write the following sets as \( X \times Y \) for some sets \( X, Y \), or \( X \times Y \times Z \) for some sets \( X, Y, Z \). If not possible, try to explain why not. It may help to draw the set.
   (a) The set of points on the top and bottom face of the unit cube \([0, 1] \times [0, 1] \times [0, 1]\)
   (b) The set of lattice points on the \( xy \)-plane
   (c) The set of horizontal gridlines on the \( xy \)-plane
   (d) The set of vertical asymptotes of \( y = \tan(x) \)
   (e) The set of points on any of the 4 sides of the square \([0, 1] \times [0, 1]\)
   (f) The set of points on the unit circle \( x^2 + y^2 = 1 \)
   (g) The set of points on any line of the form \( y = x + b \) where \( b \in \mathbb{Z} \)
   (h) The set of points on any line of the form \( y = x + b \) where \( b \in \mathbb{R} \)
Subsets, Power Sets (1.3-1.4)

1. Write the power set of each of the following sets. What is the cardinality of the power set?
   (a) \{1, 2, 3\}
   (b) \emptyset
   (c) \{\emptyset\}
   (d) \{\{1, 2, 3\}\}
   (e) \{\{1, 2\}, \{3, \{4\}\}\}
   (f) \{X\}, where \(X = \{a \in \mathbb{R} : a^2 = 3\}\)

2. Explicitly write out all elements in the following sets.
   (a) \(\{X \in \mathcal{P}(A) : |X| \leq 2\}\), where \(A = \{1, 2, 3, 4\}\)
   (b) \(\{X : X \subseteq \{1, 2, 3\}, 2 \in X\}\)
   (c) \(\{X \subseteq \mathbb{N} : |X| = 1\}\)
   (d) Is the set in the previous part the same as \(\mathbb{N}\)? Why or why not?
   (e) \(\mathcal{P}(\{1, 2\} \times \{3, 4\})\)
   (f) \(\mathcal{P}(\{1, 2\}) \times \mathcal{P}(\{3, 4\})\)
   (g) \(\mathcal{P}(\emptyset)\)
   (h) Suppose \(X\) and \(Y\) are sets. What kinds of objects are the elements of \(\mathcal{P}(X \times Y)\)? What kinds of objects are the elements of \(\mathcal{P}(X) \times \mathcal{P}(Y)\)? Can these sets ever be the same?

3. Determine the cardinality of the following sets, given that \(|A| = n\) and \(|B| = m\).
   (a) \(\mathcal{P}(A) \times \mathcal{P}(B)\)
   (b) \(\mathcal{P}(A \times B)\)
   (c) \(\mathcal{P}(A \times \mathcal{P}(A \times B))\)
   (d) \(\mathcal{P}(\mathcal{P}(A))\)
   (e) \(\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(A \times \emptyset))\)
   (f) \(\{X \in \mathcal{P}(A) : |X| = 1\}\)
   (g) \(\{X \in \mathcal{P}(A) : |X| \leq 1\}\)

4. For each of the following statements below, which of the following symbols can correctly be placed in the blank? Select all that apply.
   Possible symbols: \(\in, \not\in, \subseteq, \subset\)
   (a) \(\{1, 2\} \, \_\_\_ \{1, 2, 3\}\)
   (b) \(1, 2 \, \_\_\_ \{1, 2, 3\}\)
   (c) \(\{1, 2\} \, \_\_\_ \mathcal{P}(\{1, 2, 3\})\)
   (d) \(1, 2 \, \_\_\_ \mathcal{P}(\{1, 2, 3\})\)
   (e) \(\{1, 2\} \, \_\_\_ \{\{1, 2, 3\}\}\)
   (f) \(\{1, 2\} \, \_\_\_ \mathcal{P}(\{\{1, 2, 3\}\})\)
   (g) \(\{1, 2\} \, \_\_\_ \{\{1, 2\}, 1, 2, 3\}\)
   (h) \(\{1, 2\} \, \_\_\_ \mathcal{P}(\{\{1, 2\}, 3\})\)
Set Operations (1.5)

1. Compute the following, given \( A = \{1, 2, 3, 4, 5, 6\} \), \( B = \{1, 5, 6, 7, 8\} \), \( C = \{1, 2\} \).
   
   (a) \( A \cup B \), \( A \cup C \), and \( B \cup C \)
   
   (b) \( (A \cup B) \cup C \) and \( A \cup (B \cup C) \)
   
   (c) \( A \cap B \), \( A \cap C \), and \( B \cap C \)
   
   (d) \( (A \cap B) \cap C \) and \( A \cap (B \cap C) \)
   
   (e) \( A - B \), \( A - C \), \( B - A \), \( B - C \), \( C - A \), and \( C - B \)
   
   (f) \( (A \times C) \cap (C \times B) \)
   
   (g) \( P(A) \cap P(B) \)
   
   (h) \( P(A) \cap (B \times C) \)

2. Write the following sets as a single interval or the union of finitely many disjoint intervals. (We say two sets \( A, B \) are disjoint if \( A \cap B = \emptyset \)).
   
   (a) \([1, 2] \cup [2, 4]\)
   
   (b) \([1, 2] \cap [2, 4]\)
   
   (c) \([1, 2] - [2, 4]\)
   
   (d) \([-3, -1] \cup [-4, -2]\)
   
   (e) \([-3, -1] \cap [-4, -2]\)
   
   (f) \([-3, -1] - [-4, -2]\)
   
   (g) \([1, 4] - [2, 3]\)
   
   (h) Can you say anything about \([a, b] \cup [c, d]\), \([a, b] \cap [c, d]\), and \([a, b] - [c, d]\) in general?

3. Convert from set-builder notation into a natural choice of union or intersection of some sets.
   
   (a) \(\{x \in \mathbb{Z} : -10 \leq x \leq 10\}\)
   
   (b) \(\{x \in [-10, 10] : x \in \mathbb{Z}\}\)
   
   (c) \(\{(x, y) \in \mathbb{R}^2 : x^2 \leq 5 \text{ and } y > x\}\)
   
   (d) \(\{(x, y) \in \mathbb{R}^2 : x^2 \leq 5 \text{ or } y > x\}\)

4. True or false? If true, explain why. If false, give an example where the statement doesn’t hold. Can you fix the statement to make it true?
   
   (a) (T/F) For any two sets \( A, B \), we have \( A \cap B \in \mathcal{P}(A) \) and \( A \cap B \in \mathcal{P}(B) \).
   
   (b) (T/F) For any two sets \( A, B \), we have \( A \cup B \in \mathcal{P}(A) \) and \( A \cup B \in \mathcal{P}(B) \).
   
   (c) (T/F) For any two sets \( A, B \), we have \( A - B \in \mathcal{P}(A) \) and \( A - B \in \mathcal{P}(B) \).
   
   (d) (T/F) If \( A \subseteq B \), then \( A \cup B \subseteq A \) and \( A \cup B \subseteq B \).
   
   (e) (T/F) If \( A \subseteq B \), then \( A \times B \subseteq B \times B \).
   
   (f) (T/F) If \( A \subseteq B \), then \( (A \times B) \cap (B \times A) = A \times A \).
   
   (g) (T/F) If \( A \subseteq B \), then \( (B - A) \times A = (B \times A) - (A \times A) \)
Complements (1.6), Venn Diagrams (1.7)

1. Let $A, B, C \subseteq U$. Draw a Venn Diagram to illustrate each of the following pairs of sets. When, if ever, are these sets equal? When are they not equal? When possible, give examples of nonempty sets $A, B, C$ for which the both sets are equal, and examples of nonempty sets $A, B, C$ for which both sets are not equal.

   (a) $A \cap (B \cup C)$ and $(C \cap A) \cup (A \cap B)$
   (b) $A \cap (B \cup C)$ and $(A \cap B) \cup (B \cap C)$
   (c) $A \cap B \cap C$ and $A \cap (B \setminus C)$ (The symbol $\setminus$ is alternate notation for the set difference).
   (d) $A \cup (B \cap (C \setminus A))$ and $A \cup (B \cap C)$
   (e) $(A \cup B) \setminus (A \cap B)$ and $(A \setminus B) \cup (B \setminus A)$

   The set in the previous part is referred to as the symmetric difference between $A$ and $B$, often denoted $A \Delta B$.

   (f) Express $(A \cup (B \cap (C \setminus A^c)))$ as simply as possible. You may want to use a Venn Diagram to help.

2. Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5, 7\}$ be subsets of the universal set $U = \{1, 2, 3, 4, 5, 6, 7\}$. Compute the following.

   (a) $B^c \setminus A^c$
   (b) $B \cap (A \setminus (B^c \cap A))$
   (c) $B^c \times A$
   (d) $B^c \times A^c$
   (e) $(B \times A)^c$ (What is the universal set in this context?)

3. Draw the complements of the following subsets of $\mathbb{R}^2$. Can you write set-builder notation for the complement?

   (a) $\{(x, y) \in \mathbb{R}^2 : y \leq x^2\}$
   (b) $(0, 1] \times [0, 2)$
   (c) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \in [1, 4)\}$
Indexed Sets (1.8)

1. Compute the following.
   (a) \( \cap_{i=1}^{\infty} [0, i + 1] \)
   (b) \( \cup_{i=1}^{\infty} [i, i + 1) \)
   (c) \( \cup_{i=1}^{\infty} (i, i + 1) \)
   (d) Let \( A, B \) be two sets. Can you give a description of \( \cup_{a \in A} \{a\} \times B \)?
   (e) \( \cup_{X \in \mathcal{P}(\mathbb{N})} X \)
   (f) \( \cap_{X \in \mathcal{P}(\mathbb{N})} X \)
   (g) Let \( A \) be a nonempty set. Give a description of the following set:
      \[ \bigcup_{X \subseteq A} X \]
      Also give a description of the following set:
      \[ \bigcap_{X \subseteq A} X \]
      Give an example of a set \( A \) for which the two sets above differ.

2. For each \( n \in \mathbb{N} \), let \( A_n = (-n, n) \subseteq \mathbb{R} \).
   (a) What is \( \cap_{n=1}^{\infty} A_n \)?
   (b) What is \( \cup_{n=1}^{\infty} A_n \)?

3. Define \( A_k = \left[ \frac{1}{k}, 1 + \frac{1}{k} \right) \) for \( k \in \mathbb{N} \).
   (a) Is \( 1 \in \cap_{k=1}^{\infty} A_k \)? What about \( 0 \)? Why or why not?
   (b) Determine \( \cap_{k=1}^{\infty} A_k \). Can you justify your claim?
   (c) Determine \( \cup_{k=1}^{\infty} A_k \). Can you justify your claim?

4. For each real number \( r \in [0, \infty) \), denote \( A_r = [0, r] \times [r + 2, \infty) \subseteq \mathbb{R}^2 \).
   (a) Illustrate \( A_r \) for a few choices of \( r \). What is \( A_0 \)?
   (b) Illustrate \( \cup_{r \in [0,2]} A_r \). Can you write this set in set-builder notation as simply as possible?
   (c) Illustrate \( \cup_{r \in [0,\infty)} A_r \). Can you write this set in set-builder notation as simply as possible?
   (d) Illustrate \( \cap_{r \in (1,2,3)} A_r \). Is there a simple description of this set?

5. For each \( (a, b) \in \mathbb{R}^2 \setminus \{0\} \), let \( S_{(a,b)} = \{ (x, y, z) \in \mathbb{R}^3 : ax + by = 0 \} \).
   (a) Draw (or describe) \( S_{(1,1)} \), \( S_{(-1,1)} \), and \( S_{(0,1)} \).
   (b) Give an example of a point \( (x, y, z) \in \cap_{(a,b) \in \mathbb{R}^2 \setminus \{0\}} S_{(a,b)} \). Can you find more? Can you determine the intersection entirely?
Logical Statements (2.1), And/Or/Not (2.2)

These exercises also cover sections 2.5-2.6 (truth tables and logical equivalence).

1. Two expressions are **logically equivalent** if they always have the same truth values. For instance, \( P \) and \( \neg \neg P \) are logically equivalent.

Let \( P, Q, R \) be statements. For each set of four expressions below, pair up the logically equivalent ones. Justify they are logically equivalent by using truth tables.

(a) \( \neg(P \lor Q) \), \( \neg(P \lor Q) \), \( \neg P \land \neg Q \), \( \neg P \lor \neg Q \)

(The equivalences you will find here are known as **DeMorgan’s Laws**).

(b) \( P \land (Q \lor R) \), \( (P \lor Q) \land (P \lor R) \), \( P \lor (Q \land R) \), \( (P \land Q) \lor (P \land R) \)

(The equivalences you find here may be referred to as **distributive laws**.)

2. Suppose \( P \) and \( Q \) are statements. Create truth tables for each of the following statements, and determine an easier logically equivalent expression.

(a) \( P \lor (P \land Q) \)

(b) \( P \land (P \lor Q) \)

(c) \( \neg(\neg P \lor (\neg Q \land P)) \)

3. A **tautology** is an expression that is always true. For instance, \( P \lor \neg P \) is a tautology. Similarly, a **contradiction** is an expression that is always false. For instance, \( P \land \neg P \) is a contradiction.

Let \( P, Q, R \) be statements. Determine if the following expressions are tautologies, contradictions, or neither. If the expression is neither a tautology nor a contradiction, give truth values for \( P, Q, R \) which make the expression true, and also give truth values for \( P, Q, R \) which make the expression false.

(a) \( \neg(P \land \neg P) \)

(b) \( P \land \neg((Q \land \neg P)) \)

(c) \( P \land (Q \lor R) \)

(d) \( P \land ((Q \land (\neg P \land \neg R)) \lor (R \land \neg P)) \)

(e) \( P \land ((Q \land (\neg Q \lor \neg R)) \lor (R \land \neg P)) \)

4. Write the statements below using the statements \( P, Q, R \) as well as combinations of \( \land, \lor, \) and \( \neg \).

(a) Exactly one of \( P \) and \( Q \) is true.

(b) Exactly one of \( P, Q, \) and \( R \) is true.

(c) Exactly two of \( P, Q, \) and \( R \) are true.

(d) Either \( P \) is true, or exactly one of \( Q \) and \( R \) is true.

(As is always the case in math, ’or’ in this sentence means inclusive or.)

5. Let \( A, B, C \subseteq U \) be subsets of universal set \( U \). For each \( x \in U \), let \( P(x), Q(x), \) and \( R(x) \) be the statements \( x \in A, x \in B, \) and \( x \in C \) respectively. Rewrite the following sets using unions, intersections, set differences, symmetric set differences, and complements. (**Hint:** It might help to replace \( P(x), Q(x), R(x) \) with their corresponding statements.)

(a) \( \{x \in U : P(x)\} \)

(b) \( \{x \in U : P(x) \land Q(x)\} \)

(c) \( \{x \in U : P(x) \land Q(x) \land \neg R(x)\} \)

(d) \( \{x \in U : (P(x) \lor Q(x)) \lor \neg P(x) \lor \neg Q(x)\} \)

(e) Write \( (A \cap B)^c \), \( A^c \cap B^c \), \( (A \cup B)^c \), \( A^c \cup B^c \) each using set builder notation like above. Which of these sets, if any, are the same? Explain.
Conditional Statements (2.3), Biconditional Statements (2.4)

1. Use truth tables to determine which of pairs of statements are logically equivalent.
   (a)  $P \Rightarrow Q$, $Q \Rightarrow P$, $P \lor \neg Q$, $Q \lor \neg P$
   (b)  $P \Rightarrow Q$, $Q \Rightarrow P$, $\neg P \Rightarrow \neg Q$, $\neg Q \Rightarrow \neg P$

   $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$. Is $P \Rightarrow Q$ equivalent to its converse?

   (c)  $P \iff Q$, $\neg P \iff \neg Q$, $\neg Q \Rightarrow \neg P$, $P \Rightarrow Q$

   $\neg Q \Rightarrow \neg P$ is called the **contrapositive** of $P \Rightarrow Q$. Is $P \Rightarrow Q$ equivalent to its contrapositive?

   (d)  $(P \Rightarrow Q) \land (Q \Rightarrow P)$, $(\neg Q \Rightarrow \neg P) \land (P \Rightarrow Q)$, $(\neg Q \Rightarrow \neg P) \lor (P \Rightarrow Q)$, $(P \Rightarrow Q) \land (\neg P \Rightarrow \neg Q)$

   (e)  $P \Rightarrow Q$, the converse of the contrapositive of $P \Rightarrow Q$, the contrapositive of the converse of $P \Rightarrow Q$, the converse of the contrapositive of the converse of $P \Rightarrow Q$

2. For each of the following, determine if the statement is of the form $P \Rightarrow Q$, $Q \Rightarrow P$, $P \iff Q$, or none of the above. (When stuck, try truth tables.)
   (a)  $Q$ holds if $P$ does
   (b)  $Q$ holds only if $P$ does
   (c)  $P$ holds whenever $Q$ does
   (d)  $P$ does not hold whenever $Q$ does not hold
   (e)  For $Q$ to hold, it is sufficient for $P$ to hold.
   (f)  For $Q$ to hold, it is necessary that $P$ must also hold.
   (g)  Either $P$ and $Q$ both hold or neither does
   (h)  Neither $P$ nor $Q$ hold
   (i)  Either $Q$ holds or $P$ doesn’t hold

3. Use truth tables and/or rules from problem 1 in order to replace the following expressions with simpler logically equivalent expressions. If it is a tautology or contradiction, you may simply write "tautology" or "contradiction".
   (a)  $P \land (P \Rightarrow Q)$
   (b)  $Q \land (P \Rightarrow Q)$
   (c)  $P \lor (P \Rightarrow Q)$
   (d)  $Q \lor (P \Rightarrow Q)$
   (e)  $(P \land Q) \Rightarrow Q$
   (f)  $(P \Rightarrow Q) \land (Q \Rightarrow Q)$ (Is this equivalent to the previous part?)
   (g)  $(P \lor Q) \Rightarrow Q$
   (h)  $(Q \Rightarrow \neg P) \lor (P \Rightarrow \neg P)$

4. Let $A, B \subseteq U$ be subsets of universal set $U$. Let $P(x)$ denote the expression "$x \in A$", and let $Q(x)$ denote the expression "$x \in B$". Determine the following sets in terms of $A, B$, unions, intersections and complements.
   (a)  $\{x \in U : P(x) \Rightarrow Q(x)\}$
   (b)  $\{x \in U : Q(x) \Rightarrow P(x)\}$
   (c)  $\{x \in U : P(x) \iff Q(x)\}$
Quantifiers (2.7)

1. Rewrite the following using quantifiers as appropriate. Then, decide if the statement is true or false.
   (a) The function \( f(x) = \sin(x) \) sometimes has negative output.
   (b) The function \( f(x) = x^2 + 1 \) is always positive.
   (c) The function \( f(x) = \sin(x) \) is always positive on the domain \((0, \pi)\).
   (d) The set \( A = \{ r^2 : r \in \mathbb{Z} \} \) only consists of positive real numbers.
   (e) There are integers \( a \) and \( b \) for which \( \sqrt{2} \) can be written as \( a/b \).
   (f) Can you give an expression involving quantifiers that is logically equivalent to the expression ”\( x \in \mathbb{Q} \)”?

2. The following statements use quantifiers. Write a sentence to try to parse what the statement is saying. Then, decide if the statement is true or false.
   (a) \( \exists a \in \mathbb{R} \) such that \( \forall x \in \mathbb{Z}, a + x = x \).
   (b) \( \forall z \in \mathbb{R}, \text{if } z \in \mathbb{Q}, \text{then } \sin(z) \notin \mathbb{Q} \).
   (c) \( \forall N \in \mathbb{Z}, \exists k \in \mathbb{N} \text{ with } k^2 > N \).
   (d) \( \exists X, Y \in \mathcal{P}(\mathbb{R}) \text{ with } X \times Y = \{(1,2), (3,4), (5,6), (7,8)\} \).
   (e) \( \exists X, Y \in \mathcal{P}(\mathbb{R}) \text{ with } |X \times Y| = 13 \).
   (f) \( \forall \epsilon > 0, \text{ there exists an integer } k \in \mathbb{Z} \text{ such that } \frac{1}{k} < 1 - \epsilon \).
   (g) \( \forall \epsilon > 0 \text{ and } \forall x \in \mathbb{R}, \exists \delta > 0 \text{ such that if } |x - 1| < \delta, \text{ then } |x^2 - 1| < \epsilon \).

3. Let \( A, B \subseteq U \) be subsets of universal set \( U \). For each statement below, write an equivalent statement that involves quantifiers. If possible, decide if the statement is true or false in general. Otherwise, give examples where it’s true and examples where it’s false.
   (a) \( A \cap B^c \subseteq B \)
   (b) \( A - B = \emptyset \)
   (c) \( A - B \neq \emptyset \)
   (d) \( A = B \)
   (e) \( A \neq B \)
   (f) \( A \neq B \iff A - B \neq \emptyset \)
Negation (2.10)

1. Negate each of the following statements. Decide, if possible, whether the original statement is true, or its negation is true.
   
   (a) \( \exists x, y \in \mathbb{R} \text{ s.t. } y \leq x \text{ or } x < y \)
   
   (b) \( \forall x, y \in \mathbb{R}, y \leq x \text{ or } x < y \)
   
   (c) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } y < x \)
   
   (d) \( \exists y \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{R}, y < x \) (Is this different from the previous statement?)

2. Let \( A, B \subseteq \mathbb{R} \) be be subsets of the real numbers. Negate the following statements. For each, give an example of sets \( A \) and/or \( B \) where the original statement is true, and an example where the negation is true.
   
   (a) \( \forall a, b \in A, a + b \in A. \)
   
   (b) \( \forall a \in A \text{ and } \forall b \in B, a \leq b \)
   
   (c) \( \exists a \in A \text{ s.t. } \forall b \in B, a \leq b \)
   
   (d) \( \forall \beta \in B, \beta < \beta - 1 \)

3. Consider the following statements from calculus. Do you recognize them? What are their negations?
   
   (a) \( \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |a_n - L| < \epsilon \)
   
   (b) \( \forall M > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, a_n > M \)
   
   (c) \( \forall \epsilon > 0, \exists \delta > 0 \text{ such that for all } x \in \mathbb{R}, |x - a| < \delta \text{ implies } |f(x) - f(a)| < \epsilon \)
   
   (d) Show the open interval \((0, 1)\) has the following property: \( \forall x \in (0, 1), \exists \epsilon > 0 \text{ such that } 0 < x - \epsilon < x + \epsilon < 1. \)

4. Negate each of the following sentences. It will help to analyze the logical structure of the original sentence.
   
   (a) Every person named Matthew can eat gluten.
   
   (b) There are no people named Bob that remember everything that they have ever said.
   
   (c) Given three menu items to choose from, I never choose what I chose day ago or two days ago.
   
   (d) If \( p \) is a polynomial with real coefficients and \( p' \) is constant, then \( p \) has degree 1.
   
   (e) There are not any students in this class that cannot count the number of knot theory classes every non-European math university has never taught.
   
   (f) Once upon a time, in every kingdom, there was a tower that no one could climb.
   
   (g) In every kingdom, there was a tower that once upon a time no one could climb.
   
   (h) Either I will not study for this midterm, or I will not do well on it, or both
Midterm 1 Review Questions

1. Write set-builder notation to define the interval \([a, b]\). How do you deal with \((a, b), (a, b], [a, b)\)? What about \([a, \infty), (a, \infty), (-\infty, b), (-\infty, b]\)? Is \(\infty \in (0, \infty)\)? Why or why not?

2. Are any of the sets \([-1, 1], (-1, 1), \{-1, 1\}, [-1, 1], (1, -1), (-1, 1)\) the same? Is any of the notation wrong here? How many elements does each set have? Which, if any, of the sets only contain integers? What kind of elements are in \([-1, 1]\)? (Sets? Lines? Something else?)

3. Write the following sets as unions of as few disjoint intervals (or singletons) as possible: \((1, 3) \setminus [0, 2], [1, 3) \setminus (1, 2), [2, 4] \setminus \{3\}, [1, 2)^c\).

4. Let \(A, B \subseteq U\) be subsets of universal set \(U\). When is an element \(x\) in the set \(A \cup B\)? Your answer should be something of the form \(x \in A \cup B\) if and only if _______. Similarly answer this question for \(A \cap B, A \setminus B, B \setminus A, A^c\), and anything else you can think of. Use this phrasing and logic rules that you know to argue \(A \cap (A \cup B) = A\). What about \((A \cup B) \setminus A, (A \cup B) \setminus (A \cap B), (A \cap B) \setminus B\)? Use Venn Diagrams, but also logic rules. Can you come up with other statements about sets using other logic rules that you know?

5. For each \(k \in \mathbb{N}\), let \(R_k = \{n \in \mathbb{N} : \sqrt{kn} \in \mathbb{Q}\}\). What is \(R_1\)? \(R_2\)? \(R_4\)? \(R_9\)? \(R_{10}\)? Describe the set \(S = \{j \in \mathbb{N} : 3 \in R_j\}\). What kinds of elements are in this set? Integers? Real numbers? Sets? Lines? Something else? Can you write down all the elements of \(S\)?

6. What cardinalities, if any, are impossible for the Cartesian product of two sets? What cardinalities, if any, are impossible for the power set of a set? Give examples of sets \(A\) and \(B\) with \(A \cup B\) and \(A \cap B\) having the same number of elements. How might the cardinalities of \(A, B, A^c, B^c, A \setminus B, B \setminus A, A \cap B, A \cup B\) compare in general? Consider cases where the cardinalities are finite and cases where the cardinalities of some or all of these sets are infinite.

7. Give an example of a tautology, and an example of a contradiction. What are some laws associated to tautologies and contradictions? Use these and other logic laws to show \((Q \vee \neg P) \land (R \lor \neg Q) \land (P \lor \neg R)\) is logically equivalent to \(((P \land Q) \lor (\neg P \land \neg Q)) \land ((P \lor R) \lor (\neg P \land \neg R))\). (You don’t need to write out every term. You can expand the first to get 8 terms, but all but two will ‘cancel’ in the sense that they’ll give contradictions. Similarly, the second will give 4 terms, for which all but two will contradictions.)

8. Use truth tables find a simpler expression for \((\neg P \lor R) \land (P \lor R)\). Is there a different way to see this simplification? (Consider the distributive law).

9. John says 8 \(\in [0, 3] \times [1, 7]\), because 2 \(\in [0, 3]\), 4 \(\in [1, 7]\), and 2 \(\times 4 = 8\). Is he right?

10. What kinds of elements are in \([0, 3] \times [1, 7]\)? Sets? Lines? Numbers? Rectangles? Something else?

11. Draw \(\mathbb{Z} \times [0, 1], \mathbb{Z} \times \mathbb{R}, \mathbb{Z} \times \mathbb{Z}, [0, 1] \times \{0, 1\}, \) and \((0, 1) \times \{0, 1\}\). In general, describe how we decide whether \((x, y)\) is in a Cartesian product or not. For example, for \((0, 1) \times \{3, 4\}\), which of the corners \((0, 3), (1, 3), (0, 4), (1, 4)\) are in the set? Which are not? Is there anything else in the set? Draw it, and draw its complement.

12. True or false: the negation of "For every doctor, there exists an apple which keeps the doctor away" is "For every doctor, there does not exist an apple which keeps the doctor away." If you answer true, explain. If you answer false, state what you think the negation is, and explain why it is different from what is written here. What is the negation of "For every doctor the size of an apple, there exists an apple the size of a car that keeps that doctor away."? Which is true, the original statement or its negation?

13. What kinds of elements are in \(\mathcal{P}([0, 1])\)? What kinds of elements are in \(\{0, 1\}\)? What about \(\mathcal{P}([0, 1])\)? What about \(\mathcal{P}([0, 1] \times \{0, 1\})\), or \(\mathcal{P}([0, 1]) \times \mathcal{P}([0, 1])\)? Are any of these sets infinite?
Direct Proofs (4.3)

We say an integer \( n \in \mathbb{Z} \) is **even** if there exists an integer \( k \in \mathbb{Z} \) with \( n = 2k \).

We say an integer \( n \in \mathbb{Z} \) is **odd** if there exists an integer \( k \in \mathbb{Z} \) with \( n = 2k + 1 \).

For now, we’ll use the following facts: *every integer is either even or odd, but never both.*

1. Let \( x, y \in \mathbb{Z} \). Prove that if \( x \) and \( y \) are odd, then \( xy \) is odd and \( x + y \) is even. State similar facts for sums and products of even and odd integers, though you do not need to prove them.

2. Let \( x \in \mathbb{Z} \) be an integer. Prove that if \( x \) is an odd integer, then \( x^2 + 3x + 5 \) is also an odd integer.

3. Show that every odd integer is the difference of two perfect squares.

4. Show that if the difference of two perfect squares is even, so is their sum.

For integers \( d, n \in \mathbb{Z} \), we say \( d \mid n \) (read "\( d \) divides \( n \)") if there exists an integer \( k \in \mathbb{Z} \) with \( dk = n \).

We write \( d \not\mid n \) for \( d \) does not divide \( n \).

3. Does \( 0 \mid 2 \)? Does \( 2 \mid 0 \)? Does \( 3 \mid 5 \)?

4. What is the set \( \{ n \in \mathbb{N} : 2 \mid n \} \)? What is the set \( \{ n \in \mathbb{N} : n \not\mid 2 \} \)?

5. Let \( a \in \mathbb{Z} \). Prove that if \( 5 \mid (2a) \), then \( 5 \mid a \).

6. Let \( a, b, c \in \mathbb{Z} \). Prove that if \( a \mid b \) and \( b \mid c \), then \( a \mid c \).

7. Let \( a, b \in \mathbb{Z} \) and \( n \in \mathbb{N} \). Prove that if \( a \mid b \), then \( a^n \mid b^n \).

8. Let \( a, b \in \mathbb{Z} \). Is it true that if \( a^2 \mid b^2 \), then \( a \mid b \)? If so, prove it. If not, give a counterexample.

9. Let \( a, b, c \in \mathbb{Z} \). Is it true that if \( a \mid b \) and \( a \mid c \), then \( a \mid (bc) \)? If so, prove it. If not, provide a counterexample.

10. Let \( a, b, c \in \mathbb{Z} \). Is it true that if \( a \mid c \) and \( b \mid c \), then \( (ab) \mid c \)? If so, prove it. If not, provide a counterexample.

11. Let \( a, b, c, d \in \mathbb{Z} \). Is it true that if \( a \mid c \) and \( b \mid d \), then \( (ab) \mid (cd) \)? If so, prove it. If not, provide a counterexample.

Well, maybe not *that* direct...

12. For \( n \in \mathbb{Z} \), \( d \in \mathbb{N} \), and \( 0 \leq r < d \), we say \( n \) leaves a remainder of \( r \) when divided by \( d \) if \( d \mid (n - r) \). Show that if \( a, b \in \mathbb{Z} \) leave a remainder of 1 when divided by \( d > 1 \), then \( ab \) also leaves a remainder of 1, and \( a + b \) leaves a remainder of 2 if \( d > 2 \) or 0 if \( d = 2 \).

13. Show that if \( a, b, c \in \mathbb{Z} \) have \( a^2 + b^2 = c^2 \), then either \( a \) is even or \( b \) is even.

14. Prove that if \( n \in \mathbb{Z} \), then \( 5n^2 + 3n + 7 \) is odd.

15. Prove that for \( n \in \mathbb{N} \), if \( 3 \mid n \) and \( 8 \mid n \), then \( 24 \mid n \). Contrast this with problem 5.

16. Prove that for \( k \in \mathbb{N} \), if \( k \) is odd and not a multiple of 3, then \( 24 \mid (k^2 - 1) \).
Casework (4.4-4.5)

1. Recall for a real number \( x \in \mathbb{R} \), we have

\[
|x| = \begin{cases} 
  x & x \geq 0 \\
  -x & x < 0 
\end{cases}
\]

Show that for any real numbers \( a, b \in \mathbb{R} \), \( |ab| = |a| \cdot |b| \).

2. Show that there are no integers \( x, y \in \mathbb{Z} \) with \( x^2 + y^2 = 3 \).

3. Show that for any \( n \in \mathbb{N} \), \( n^2 + n \) is even, and hence \( \frac{n^2 + n}{2} \) is an integer.

4. Show that for any \( n \in \mathbb{N} \), either \( 1 + (-1)^n \cdot (3n - 1) \) or \( 2 + (-1)^n \cdot (3n - 1) \) is a multiple of 6.

5. (Optional/Challenge) Show that for any \( n \in \mathbb{N} \), \( n^2 + \left(-1\right)^{n} \cdot \left(3n^2 + n\right) \cdot n^3 \) is a multiple of 4.

Proving Conditionals (4.3, 5.1, 6.2)

1. Let \( x \in \mathbb{Z} \). Show that if \( x^3 - 1 \) is even, then \( x \) is odd.

2. Let \( x, y \in \mathbb{Z} \). Show that if \( x^3(y + 3) \) is even, then either \( x \) is even or \( y \) is odd.

3. Let \( a, b \in \mathbb{R} \). Show that if \( a \in \mathbb{Q} \) and \( ab \notin \mathbb{Q} \), then \( b \notin \mathbb{Q} \).

4. Let \( a, b \in \mathbb{Z} \). Show that if \( 3|(ab) \), then either \( 3|a \) or \( 3|b \).

5. (Optional/Challenge) Let \( x, y, z \in \mathbb{Z} \). Show that if \( x^2 + y^2 = 3z^2 \), then \( 3|x \) and \( 3|y \).
Proofs by Contradiction (6.1-6.3)

1. Prove that \( \sqrt[3]{2} \) is irrational.

2. Prove that \( \log_2(3) \) is irrational.

3. Analyze whether the following argument is correct or incorrect. If it is correct, explain why. If it is incorrect, correct it.

**Claim:** \( \sqrt[3]{4} \not\in \mathbb{Q} \)

**Proof:** Suppose \( \sqrt[3]{4} \in \mathbb{Q} \). Then \( \sqrt[3]{4} \) is rational. But \( \sqrt[3]{2} \) is irrational by problem 2, and squaring that gives \( \sqrt[3]{4} \). By contradiction, we get \( \sqrt[3]{4} \not\in \mathbb{Q} \). \( \square \)

4. What can you say about the sum of two rationals? What about the sum of irrationals? What about the sum of a rational and an irrational? Prove each of your claims.

5. Answer the above but for products instead of sums.

6. True or false? For any \( r \in \mathbb{R} \), \( r \log_2(3) \not\in \mathbb{Q} \)

Equivalent Statements (7.1-7.2)

1. Prove that for \( a, b \in \mathbb{N} \), we have \( a^2 + b^2 = 25 \iff a = 3, b = 4 \) or \( a = 4, b = 3 \).

2. Tim reads the previous line and notices that for both of the possible solutions for \( a \) and \( b \), we have \( ab = 12 \). Thus, Tim claims that for \( a, b \in \mathbb{N} \), \( a^2 + b^2 = 25 \iff ab = 12 \). Is this claim correct? If so, prove it. If not, is either implication correct?

3. For this problem, it will be helpful to draw something!

Lisa has to prove that statements \( A, B, C, D \) and \( E \) are equivalent.

(a) If she’s already proved \( A \implies B, B \implies C, B \implies D, D \implies A, D \implies E, \) and \( E \implies B \), what’s the least number of implications she has left to prove?

(b) If she’s already proved \( A \iff B, B \iff C \iff D, A \iff E \iff C, \) and \( \neg C \iff \neg D \), what’s the least number of implications she has left to prove?

4. Let \( r \in \mathbb{R} \). Prove \( r \log_2(3) \in \mathbb{Q} \iff r = 0 \).

5. Let \( a \in \mathbb{N} \). Show \( a^3 | a \iff a^2 = 1 \iff a = 1 \).

6. Let \( a \in \mathbb{N} \). Show the following are equivalent:
   
   (a) \( 100 | a \)
   
   (b) \( 4 | a \) and \( 25 | a \)
   
   (c) \( 4 | a \) and \( 50 | a \)

7. Let \( a \in \mathbb{N} \). Show that \( a | 10 \iff a | 210 \) and \( a | 20 \).

8. (Optional/Challenge) Let \( a, b, c \in \mathbb{N} \). Can you describe which \( d \in \mathbb{N} \) has the following property?

\[ a | d \iff a | b \text{ and } a | c \]

Similarly, can you describe which \( e \in \mathbb{N} \) has the following property?

\[ e | a \iff b | a \text{ and } c | a \]