# A Limit Law for the Maximum of Subcritical DG-Model on a Hierarchical Lattice

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# Discrete Gaussian (DG) model on $\mathbb{Z}^d$

## Definition

The DG-model in $\Lambda \subset \mathbb{Z}^d$ is an integer-valued random field $\{h_x\}_{x\in\mathbb{Z}^d}$ with law

$$P(dh) = \frac{1}{Z_{\Lambda,\beta}} \mathrm{e}^{-\beta H(h)} \prod_{x\in\Lambda} \#(\mathrm{d}h_x) \prod_{x\notin\Lambda} \delta_0(\mathrm{d}h_x)$$

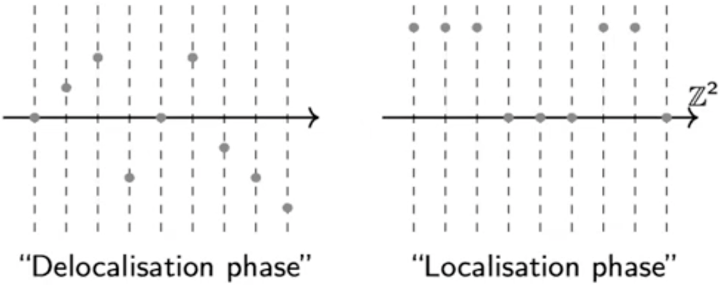
where

- $\beta > 0$ inverse temperature
- $H(h) := \frac{1}{2}\sum_{x\sim y}(h_x - h_y)^2$
- $\#$ counting measure on $\mathbb{Z}$
- $Z_{\Lambda,\beta}$ normalization constant

- Also known as integer-valued DGFF or $\mathbb{Z}$-Ferromagnet

# Phase transition of the DG-model in $\mathbb{Z}^2$

$\beta \ll 1$ (high temperature) $\beta \gg 1$ (low temperature)



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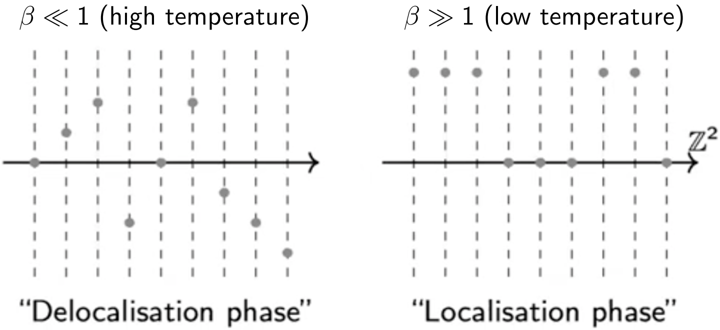


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$$E(h_x^2) = O(1)$$

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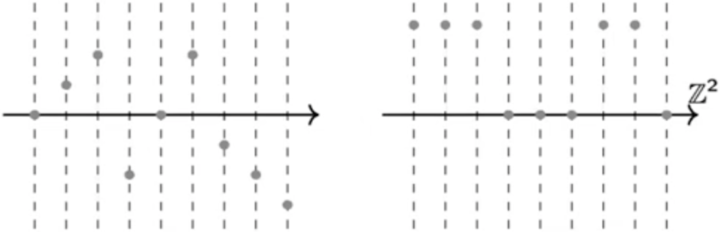


"Delocalisation phase" "Localisation phase"

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- [Fröhlich & Spencer 1981] For $\beta \ll 1$,

$$E(h_x^2) \asymp \log L$$

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[Aizenman, Harel, Peled, Shapiro 2022] $\exists\, \beta_c(\mathbb{Z}^2) \in (0, \infty)$ such that as $L \to \infty$,

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## Asympotics of the maximum

- [Lubetzky, Martinelli, Sly 2014] For $\beta \gg 1$, $\exists\, N = N(L)$ such that

$$\max_{x \in \Lambda_L} h_x \in \{N, N+1\} \quad \text{with} \quad N(L) \sim \sqrt{(2\pi\beta)^{-1} \log L \log\log L}$$

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- [Wirth 2019] For $\beta \ll 1$,

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# Extremal properties of log-correlated random fields

For $\varphi =$ a log-correlated random field (examples given on the next slide), set

$$M_n := \max_{x \in \Lambda_n} \varphi_x$$

- $\exists c_1, c_2 > 0$ such that for $m_n := c_1 n - c_2 \log n$, $\{M_n - m_n\}_{n \geq 1}$ is tight

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- $\exists c_* > 0$, $\alpha > 0$, and $\mathcal{Z}$ a.s. positive random variable such that

$$\forall t \in \mathbb{R}: \qquad P\left(M_n \leq m_n + t\right) \underset{n\to\infty}{\longrightarrow} E\left(\mathrm{e}^{-c_* \mathcal{Z} \mathrm{e}^{-\alpha t}}\right)$$

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- $\exists$ a law $\mathcal{D}$ on locally-finite upper-bounded point processes on $\mathbb{R}$ such that

$$\sum_{x\in\Lambda_n} \delta_{\varphi_x - m_n} \xrightarrow[n\to\infty]{\text{law}} \sum_{i,j\geq 1} \delta_{h_i + d_j^{(i)}},$$

where $\{h_i : i \geq 1\}$ enumerates the points in a sample from

$$\mathrm{PPP}\Big(\mathcal{Z}\mathrm{e}^{-\alpha h}\,\mathrm{d}h\Big),$$

and $\{d_j^{(i)} : j \geq 1\}_{i\geq 1}$ enumerates the points in the $i$-th member of the sequence $\{d^{(i)}\}_{i\geq 1}$ of i.i.d. samples from $\mathcal{D}$, called **decorations**, independent from $\{h_i : i \geq 1\}$ and $\mathcal{Z}$

# Extremal properties of log-correlated random fields

The general picture above holds for examples including:

- Branching Brownian Motion at time $n$
- Branching Random Walk (BRW) at time $n$
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Further evidence of universality:

- Four-dimensional membrane model
- Local time of simple random walk on a regular tree
- Characteristic polynomial of a random matrix ensemble
- A class of $P(\varphi)_2$-models on a torus

# Hierarchical Lattice

Let $n \geq 0$ and $b \geq 2$ be integers. The hierarchical lattice of depth $n$ is defined by

$$\Lambda_n = \{1, \ldots, b\}^n$$

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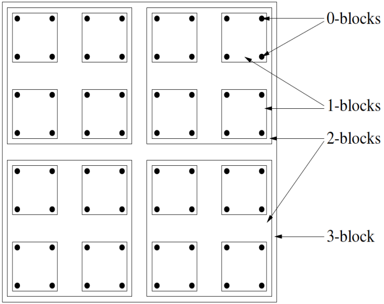
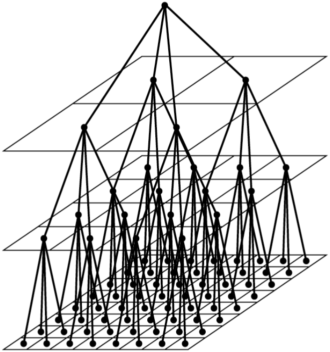



Blocks in $\mathcal{B}_j$ for $j = 0, 1, 2, 3$ when $d = 2$, $N = 3$, $L = 2$.



# Hierarchical Laplacian $\Delta_n$

For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \Lambda_n$, the hierarchical distance between $x$ and $y$ is defined by

$$d(x, y) := \inf\{j \in \{0, \ldots, n\} : x_i = y_i \ \forall i = 1, \ldots, n - j\}$$

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$\Delta_n$ acting on $f : \Lambda_n \to \mathbb{R}$ takes the explicit form

$$\Delta_n f(x) = \sum_{k=1}^{n} (\lambda_{k-1} - \lambda_k) b^{-k} \sum_{y \in \mathcal{B}_k(x)} [f(y) - f(x)] - \lambda_n f(x),$$

where $\lambda_k := \left(\sum_{j=0}^{k} b^j\right)^{-1}$ and $\mathcal{B}_k(x) := \{y \in \Lambda_n : d(x, y) \le k\}$

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- $\Delta_n$ is the generator of a Markov chain induced by the projection of simple random walk on a $b$-ary tree of depth $n$ on the leaves of tree (i.e. $\Lambda_n$), killed upon exiting the tree through the root
- $(-\Delta_n)^{-1}$ is the covariance matrix of a BRW indexed by a $b$-ary tree of depth $n$ with step distribution $\mathcal{N}(0, 1/\beta)$

# Hierarchical DG-model

## Definition

The hierarchical DG-model on $\Lambda_n$ is an integer-valued random field $\{\varphi_x\}_{x\in\Lambda_n}$ with law

$$P_{n,\beta}(d\varphi) = \frac{1}{\Sigma_n(\beta)} \mathrm{e}^{\frac{1}{2}\beta(\varphi,\Delta_n\varphi)} \prod_{x\in\Lambda_n} \#(\mathrm{d}\varphi_x)$$

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- $(\cdot,\cdot)$ canonical inner product in $\ell^2(\Lambda_n)$
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## Remark

*Replacing the counting measure by the Lebesgue measure above, we get the Hierarchical GFF, which in this case is simply a Gaussian BRW on a b-ary tree with step distribution $\mathcal{N}(0, 1/\beta)$ of depth n*

# Limit law of the maximum of the hierarchical DG-model

Let $\beta_c := \frac{2\pi^2}{\log b}$ and $\alpha := \sqrt{2\log b}$. Given $\beta > 0$, define

$$m_n := \frac{1}{\sqrt{\beta}}\left[\sqrt{2\log b}\, n - \frac{3}{2}\frac{1}{\sqrt{2\log b}}\log n\right]$$

**Theorem (Biskup, H. 2023)**

*$\exists \mathcal{Z}$ a.s. positive random variable, and $\forall\, \beta \in (0, \beta_c)$, $\exists\, \hat{c}_\beta(0) > 0$ such that $\forall\, \beta \in (0, \beta_c)$ and all increasing sequences $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers for which $s := \lim_{k\to\infty}(m_{n_k} - \lfloor m_{n_k}\rfloor)$ exists,*

$$P_{n_k,\beta}\Big(\max_{x\in\Lambda_{n_k}} \varphi_x \le \lfloor m_{n_k}\rfloor + u\Big) \underset{k\to\infty}{\longrightarrow} E\Big(\mathrm{e}^{-\hat{c}_\beta(s)\mathcal{Z}\mathrm{e}^{-\alpha\sqrt{\beta}\,u}}\Big), \quad u \in \mathbb{Z}$$

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*where $\hat{c}_\beta(s) := \hat{c}_\beta(0)\,\mathrm{e}^{\alpha\sqrt{\beta}\,s}$*

For Gaussian BRW with step distribution $\mathcal{N}(0, 1/\beta)$, the above conclusion holds with $m_n$ replacing $\lfloor m_n \rfloor$, and $(\alpha\sqrt{\beta})^{-1}$ replacing $\hat{c}_\beta(s)$

# Extremal process of the hierarchical DG-model

Consider the extremal process of the hierarchical DG-model on $\Lambda_n$

$$\eta_n := \sum_{x \in \Lambda_n} \delta_{[x]_n} \otimes \delta_{\varphi_x - \lfloor m_n \rfloor},$$

where

$$[x]_n := \sum_{i=1}^{n} (x_i - 1) b^{-i}$$

embeds $\Lambda_n$ into $[0, 1]$

# Extremal process of the hierarchical DG-model

## Theorem (Biskup, H. 2023)

*$\exists Z$ a.s.-finite random Borel measure on $[0,1]$ with $Z(A) > 0$ a.s. for all nonempty open $A \subseteq [0,1]$ and, $\forall \beta \in (0, \beta_c)$, $\exists\, \nu_\beta$ a probability measure on locally finite point processes on $\mathbb{Z}$ such that $\forall \beta \in (0, \beta_c)$ and all increasing sequences $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers for which $s := \lim_{k\to\infty}(m_{n_k} - \lfloor m_{n_k} \rfloor)$ exists,*

$$\eta_{n_k} \xrightarrow[k\to\infty]{\text{law}} \sum_{i,j\geq 1} \delta_{x_i} \otimes \delta_{h_i + t_j^{(i)}},$$

*where $\{(x_i, h_i)\}_{i\geq 1}$ enumerates the points in a sample from*

$$\mathrm{PPP}\Big(c\, \mathrm{e}^{\alpha\sqrt{\beta}\, s}\, Z(\mathrm{d}x) \otimes \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\alpha\sqrt{\beta}\, n} \delta_n\Big),$$

*where $c := \alpha^{-1}(1 - \mathrm{e}^{-\alpha\sqrt{\beta}})$, and $\{t_j^{(i)}\}_{j\geq 1}$ enumerates the points in the $i$-th member of the sequence $\{t^{(i)}\}_{i\geq 1}$ of i.i.d. samples from $\nu_\beta$ that are independent of $\{(x_i, h_i)\}_{i\geq 1}$ and $Z$*

# Interpretation in terms of Laplace Transform

The conclusion of the theorem is equivalent to $\forall f \in C_c^+([0,1]\times\mathbb{R})$,

$$E\left(\exp\left\{-\sum_{x\in\Lambda_n} f([x]_n, \varphi_x - m_n)\right\}\right)$$
$$\underset{n\to\infty}{\longrightarrow} E\left(\exp\left\{-c\mathrm{e}^{\alpha\sqrt{\beta}s}\int Z(dx)\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\alpha\sqrt{\beta}n}\left(1-\mathrm{e}^{-\int f(x,n+\cdot)d\chi}\right)\nu_\beta(d\chi)\right\}\right).$$

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2. Prove the convergence of the extremal process associated with the hierarchical DG-model from the corresponding result for BRW (Madaule 2017), again with the help of the strong coupling
3. Deduce the convergence of centered maximum from the convergence of the extremal process

# Part I

$$\Delta_n f(x) = \sum_{k=1}^{n} (\lambda_{k-1} - \lambda_k) b^{-k} \sum_{y \in \mathcal{B}_k(x)} [f(y) - f(x)] - \lambda_n f(x),$$

Define the operator

$$Q_k f(x) := b^{-k} \sum_{y \in \mathcal{B}_k(x)} f(y), \quad k = 1, \ldots, n$$

and $Q_{n+1} = 0$. Note that $Q_0 = \mathrm{Id}$

- $\{Q_k\}_{k \in \{0,\ldots,n+1\}}$ are orthogonal projections in $\ell^2(\Lambda_n)$:

$$Q_k Q_\ell = Q_\ell Q_k = Q_{j \vee k}$$

- $\{Q_k - Q_{k+1}\}_{k \in \{0,\ldots,n\}}$ are orthogonal projections on orthogonal subspaces:

$$(Q_\ell - Q_{\ell+1})(Q_k - Q_{k+1}) = \delta_{k,\ell}(Q_k - Q_{k+1})$$

Writing $\Delta_n = -\sum_{k=0}^{n} \lambda_k (Q_k - Q_{k+1})$,

$$-\Delta_n^{-1} = \sum_{k=0}^{n} \lambda_k^{-1} (Q_k - Q_{k+1})$$

# ① Renormalization Group method

Let $m\colon \Lambda_n \to \Lambda_{n-1}$ be defined by

$$m(x) := (x_1, \dots, x_{n-1}) \quad \text{when} \quad x = (x_1, \dots, x_n)$$

Recall the partition function of the hierarchical DG-model

$$\Sigma_n(\beta) = \sum_{\varphi \in \mathbb{Z}^{\Lambda_n}} \mathrm{e}^{\frac{\beta}{2}(\varphi, \Delta_n \varphi)}$$

$$\Sigma_n(\beta) = \mathfrak{z}_n \int_{\mathbb{R}^{\Lambda_{n-1}}} \Bigg( \sum_{\varphi \in \mathbb{Z}^{\Lambda_n}} \prod_{x \in \Lambda_n} \mathrm{e}^{-\frac{\beta}{2}(\varphi_x - \varphi'_{m(x)})^2} \Bigg) \mathrm{e}^{\frac{\beta}{2}(\varphi', \Delta_{n-1}\varphi')_{n-1}} \prod_{z \in \Lambda_{n-1}} \mathrm{d}\varphi'_z$$

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$$= \mathfrak{z}_n \int_{\mathbb{R}^{\Lambda_{n-1}}} \prod_{z \in \Lambda_{n-1}} \mathrm{e}^{-b v_0(\varphi'_z)} \, \mathrm{e}^{\frac{\beta}{2}(\varphi', \Delta_{n-1}\varphi')_{n-1}} \prod_{z \in \Lambda_{n-1}} \mathrm{d}\varphi'_z$$

where

$$\mathrm{e}^{-v_0(z)} := \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\frac{\beta}{2}(z-n)^2}$$

Writing $\mathfrak{g}_{1/\beta}$ as the law of $\mathcal{N}(0, 1/\beta)$, we define $v_1, \ldots, v_n$ recursively via

$$\mathrm{e}^{-v_{k+1}(z)} := \int \mathrm{e}^{-bv_k(z+\zeta)} \mathfrak{g}_{1/\beta}(d\zeta).$$

We may further write

$$\Sigma_n(\beta) = \mathfrak{z}_n \mathfrak{z}_{n-1} (2\pi/\beta)^{|\Lambda_{n-1}|/2} \int_{\mathbb{R}^{\Lambda_{n-2}}} \mathrm{e}^{-\sum_{z \in \Lambda_{n-2}} bv_1(\varphi''_z)} \mathrm{e}^{\frac{\beta}{2}(\varphi'', \Delta_{n-2}\varphi'')_{n-2}} \prod_{z \in \Lambda_{n-2}} \mathrm{d}\varphi''_z$$

$$= \cdots = \mathfrak{z}_n \Big( \prod_{k=0}^{n-1} \mathfrak{z}_k (2\pi/\beta)^{|\Lambda_k|/2} \Big) \mathrm{e}^{-v_n(0)}.$$

# ① Renormalization Group method

$$\Sigma_n(\beta) = \mathfrak{z}_n \int_{\mathbb{R}^{\Lambda_{n-1}}} \left( \sum_{\varphi \in \mathbb{Z}^{\Lambda_n}} \prod_{x \in \Lambda_n} \mathrm{e}^{-\frac{\beta}{2}(\varphi_x - \varphi'_{m(x)})^2} \right) \mathrm{e}^{\frac{\beta}{2}(\varphi', \Delta_{n-1}\varphi')_{n-1}} \prod_{z \in \Lambda_{n-1}} \mathrm{d}\varphi'_z$$

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$$= \mathfrak{z}_n \int_{\mathbb{R}^{\Lambda_{n-1}}} \Bigg( \sum_{\varphi \in \mathbb{Z}^{\Lambda_n}} \prod_{x \in \Lambda_n} \underbrace{\mathrm{e}^{v_0\left(\varphi'_{m(x)}\right)} \mathrm{e}^{-\frac{\beta}{2}(\varphi_x - \varphi'_{m(x)})^2}} \Bigg)$$

$$\mathrm{e}^{\frac{\beta}{2}(\varphi', \Delta_{n-1}\varphi')_{n-1}} \prod_{z \in \Lambda_{n-1}} \mathrm{e}^{-b v_0(\varphi'_z)} \prod_{z \in \Lambda_{n-1}} \mathrm{d}\varphi'_z$$

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$$= \mathfrak{z}_n \int_{\mathbb{R}^{\Lambda_{n-1}}} \left( \sum_{\varphi \in \mathbb{Z}^{\Lambda_n}} \prod_{x \in \Lambda_n} \underbrace{\mathrm{e}^{v_0\left(\varphi'_{m(x)}\right)} \mathrm{e}^{-\frac{\beta}{2}(\varphi_x - \varphi'_{m(x)})^2}}_{\mathfrak{q}_0\left(\varphi_x - \varphi'_{m(x)} \middle| \varphi'_{m(x)}\right)} \right)$$

$$\mathrm{e}^{\frac{\beta}{2}(\varphi', \Delta_{n-1}\varphi')_{n-1}} \prod_{z \in \Lambda_{n-1}} \mathrm{e}^{-b v_0(\varphi'_z)} \prod_{z \in \Lambda_{n-1}} \mathrm{d}\varphi'_z$$

# ② Representation as tree-index Markov chain

For $\varphi \in \mathbb{R}$, let $\mathfrak{q}(\cdot|\varphi)$ be Borel probability measure on $\mathbb{R}$ given by

$$\mathfrak{q}_k(\mathrm{d}\zeta|\varphi) := \begin{cases} \mathrm{e}^{v_k(\varphi)-bv_{k-1}(\varphi+\zeta)}\,\mathfrak{g}_{1/\beta}(\mathrm{d}\zeta), & \text{if } k \geq 1, \\ \mathrm{e}^{v_0(\varphi)-\frac{\beta}{2}\zeta^2}\,\#(\varphi+\mathrm{d}\zeta), & \text{if } k = 0. \end{cases}$$

STEP 1: for $x \in \Lambda_1$, we sample $\zeta_1(x)$ according to the law

$$\mathfrak{q}_{n-1}(\mathrm{d}\zeta|0) = \mathrm{e}^{v_{n-1}(0)-bv_{n-2}(\zeta)}\,\mathfrak{g}_{1/\beta}(\mathrm{d}\zeta)$$

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STEP 2: for $x = (x_1, x_2) \in \Lambda_2$, we sample $\zeta_2(x)$ according to the law

$$\mathfrak{q}_{n-2}\left(\mathrm{d}\zeta|\zeta_1(x_1)\right) = \mathrm{e}^{v_{n-2}(\zeta_1(x_1)) - b v_{n-3}(\zeta_1(x_1)+\zeta)} \, \mathfrak{g}_{1/\beta}(\mathrm{d}\zeta)$$

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STEP 3 to $(n-1)$: similar as above

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STEP $n$: for $x = (x_1, \ldots, x_n) \in \Lambda_n$, we sample $\zeta_n(x)$ according to the law

$$q_0\left(\mathrm{d}\zeta \middle| \sum_{j=1}^{n-1} \zeta_j(m^{n-j}(x))\right) = \mathrm{e}^{v_0\left(\sum_{j=1}^{n-1} \zeta_j(m^{n-j}(x))\right) - \frac{\beta}{2}\zeta^2} \, \#\left(\sum_{j=1}^{n-1} \zeta_j(m^{n-j}(x)) + \mathrm{d}\zeta\right)$$

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## Lemma

*For $x = (x_1, \ldots, x_n) \in \Lambda_n$, if we define*

$$\varphi_n(x) := \sum_{j=1}^{n} \zeta_j(x_1, \ldots, x_j),$$

*then $\{\varphi_n(x) : x \in \Lambda_n\}$ has the law $P_{n,\beta}$ of the hierarchical DG-model*

# ③ Control of effective potentials

Earlier analysis of the iteration

$$\mathrm{e}^{-v_{k+1}(z)} := \int \mathrm{e}^{-bv_k(z+\zeta)} \mathfrak{g}_{1/\beta}(d\zeta).$$

was based on linearization of the map $v_k \mapsto v_{k+1}$ into

$$\mathcal{L}(\tilde{v}_k)(z) := \tilde{v}_{k+1}(z) := \int \tilde{v}_k(z+\zeta)\mathfrak{g}_{1/\beta}(d\zeta)$$

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Write the Fourier representation $f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inz}$,

$$\mathcal{L}f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)b\theta^{n^2}e^{2\pi inz}, \qquad \theta := \mathrm{e}^{-2\pi^2/\beta}$$

Denoting the dual $L^1$-norm by $\|f\|_1^* := \sum_{n\in\mathbb{Z}} |\hat{f}(n)|$,

$$\|\mathcal{L}(f-\hat{f}(0))\|_1^* \leq b\theta\|f-\hat{f}(0)\|_1^*$$

# ③ Control of effective potentials

$\forall\, k \geq 0$, let $\{a_k(n)\}_{n\in\mathbb{Z}}$ be defined by

$$\mathrm{e}^{-v_k(z)} = \sum_{n\in\mathbb{Z}} a_k(n)\mathrm{e}^{2\pi i n z},$$

if $a_k \in \ell^1(\mathbb{Z})\ \forall\, k \geq 0$.

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$$\mathrm{e}^{-v_{k+1}(z)} := \int \mathrm{e}^{-b v_k(z+\zeta)} \mathfrak{g}_{1/\beta}(d\zeta)$$

translates into

$$a_{k+1}(n) := \sum_{\substack{\ell_1,\dots,\ell_b\in\mathbb{Z} \\ \ell_1+\dots+\ell_b=n}} \left[\prod_{i=1}^{b} a_k(\ell_i)\right] \theta^{n^2}, \quad n \in \mathbb{Z}.$$

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If $a_0 \in \ell^1(\mathbb{Z})$, then

$$a_{k+1}(n) \leq \|a_k\|_1^b \theta^{n^2}$$

implies $a_k \in \ell^1(\mathbb{Z})\ \forall\, k \geq 1$

# ③ Control of effective potentials

For the hierarchical DG-model, $a_0(n) = \theta^{n^2}\sqrt{\frac{2\pi}{\beta}}$, where $\theta := \mathrm{e}^{-\frac{2\pi^2}{\beta}}$

Note that

- $v_0$ even implies $v_k$ even $\forall\, k \geq 1$ and $a_k(n) = a_k(-n)$
- $a_0(n) > 0 \;\forall\, n \in \mathbb{Z}$ implies $a_k(n) > 0 \;\forall\, n \in \mathbb{Z}$

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**Lemma**

$$a_k(n) \leq [(b\theta)^{k-1} bc_0]^n \theta^{n^2} a_k(0), \quad n, k \geq 1.$$

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## Lemma

$$a_k(n) \leq [(b\theta)^{k-1} b c_0]^n \theta^{n^2} a_k(0), \quad n, k \geq 1.$$

- $$\frac{a_{k+1}(n+1)}{a_{k+1}(n)} \leq b\theta^{(n+1)^2-n^2} \sup_{\ell \geq 0} \frac{a_k(\ell+1)}{a_k(\ell)}, \quad k, n \in \mathbb{N}.$$

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- Iterating,

$$\sup_{n \geq 0} \frac{a_k(n+1)}{a_k(n)} \leq (b\theta)^k \sup_{n \geq 0} \frac{a_0(n+1)}{a_0(n)}, \quad k \in \mathbb{N}.$$

# ③ Control of effective potentials

## Theorem (Subcritical Regime)

*Let* $v_0 : \mathbb{R} \to \mathbb{R}$ *be even with* $\{a_0(n)\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ *positive and* $c_0 := \sup_{n\geq 0} \frac{a_0(n+1)}{a_0(n)} < \infty$*. When* $b\theta < 1$ *(namely* $\beta < \beta_c$*),*

$$\sup_{z,z'\in[0,1]} \Big| v_{k+1}(z) - bv_k(z') \Big| \leq 8(b\theta)^k bc_0$$

*then holds* $\forall\, k \in \mathbb{N}$ *satisfying* $(b\theta)^k bc_0 \leq 1/8$

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## Theorem (Critical Regime)

*Let* $v_0 : \mathbb{R} \to \mathbb{R}$ *be even with* $\{a_0(n)\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ *positive and* $c_0 := \sup_{n\geq 0} \frac{a_0(n+1)}{a_0(n)} < \infty$. *When* $b\theta = 1$ *(namely* $\beta = \beta_c$*),* $\exists\, \gamma > 0$ *such that*

$$\sup_{z,z'\in[0,1]} \left| v_{k+1}(z) - bv_k(z') \right| \leq 8[1+\gamma k]^{-1/2} bc_0$$

*holds* $\forall\, k \in \mathbb{N}$ *satisfying* $[1+\gamma k]^{-1/2} bc_0 \leq 1/8$

# ④ Coupling: continuum-valued steps

For $x \in \Lambda_n$, denote

$$\xi_k^{\mathrm{DG}}(x) := \zeta_{n-k}\left(m^k(x)\right)$$

so that

$$\varphi_x^{\mathrm{DG}} := \sum_{k=0}^{n-1} \xi_k^{\mathrm{DG}}(x)$$

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Let $\{\xi_k^{\mathrm{GFF}}(x)\}_{k\in\{0,\dots,n-1\},x\in\Lambda_n}$ be i.i.d. $\mathcal{N}(0,1/\beta)$ so that

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defines a hierarchical GFF (Gaussian BRW)

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$$\varphi_x^{\mathrm{GFF}} := \sum_{k=0}^{n-1} \xi_k^{\mathrm{GFF}}(x)$$

defines a hierarchical GFF (Gaussian BRW) We want a tight coupling of $\xi_k^{\mathrm{DG}}(x)$ and $\xi_k^{\mathrm{GFF}}(x)$. Note the density of $\xi_k^{\mathrm{DG}}(x)$ with respect to $\mathfrak{g}_{1/\beta}$ is given by

$$\exp\left(v_k\left(\sum_{j=k+1}^{n-1} \xi_j^{\mathrm{DG}}(x)\right) - v_{k-1}\left(\cdot + \sum_{j=k+1}^{n-1} \xi_j^{\mathrm{DG}}(x)\right)\right)$$

# ④ Coupling: continuum-valued steps

## Proposition

*Let* $\mu \overset{\text{law}}{=} \mathcal{N}(0,\sigma^2)$ *with* $\sigma \in (0,\infty)$ *and let* $X$ *and* $Y$ *have laws*

$$P(X \in A) = \int_A f_1(t)\mu(\mathrm{d}t) \quad \text{and} \quad P(Y \in A) = \int_A f_2(t)\mu(\mathrm{d}t)$$

*for some measurable* $f_1, f_2 \colon \mathbb{R} \to (0,\infty)$*. Denote the associated CDFs by* $F(t) := P(X \le t)$ *and* $G(t) := P(Y \le t)$ *and let* $h \colon \mathbb{R} \to \mathbb{R}$ *be defined by*

$$h(t) := G^{-1}(F(t))$$

*Then* $h(X) \overset{\text{law}}{=} Y$ *and, if* $\|f_1\|_\infty$, $\|1/f_1\|_\infty$, $\|f_2\|_\infty$ *and* $\|1/f_2\|_\infty$ *are finite, then*

$$\big\|h(X) - X\big\|_\infty < 3\sigma\sqrt{2\log\Big(\max\big\{\|f_1\|_\infty\|1/f_2\|_\infty,\ \|f_2\|_\infty\|1/f_1\|_\infty\big\}\Big)}.$$

$(X, h(X))$ *thus defines a coupling of* $X$ *and* $Y$ *with* $X - Y$ *bounded in* $L^\infty$

# ④ Coupling: discrete-valued step

## Lemma

*$\forall\, \beta > 0, \exists\, C_0 > 0$ such that the following holds $\forall\, z \in \mathbb{R}$: If $X \overset{\text{law}}{=} \mathcal{N}(0, 1/\beta)$ and $Y$ takes values in $z + \mathbb{Z}$ with probability*

$$P\big(Y = z + n\big) = \mathrm{e}^{v_0(z) - \frac{\beta}{2}(z+n)^2}, \quad n \in \mathbb{Z},$$

*then $\exists$ a coupling of $X$ and $Y$ such that*

$$\|X - Y\|_\infty \leq C_0$$

# ④ Coupling of the DG-model with Gaussian BRW

## Theorem (Coupling)

*$\forall\,\beta \in (0, \beta_c), \exists\, C > 0$, and $\{R_k\}_{k\geq 1}$ positive with $\limsup_{k\to\infty} k^{-1}\log R_k < 0$ and, $\forall n \geq 1$, $\exists$ a coupling of*

$$\{\xi_k^{\mathrm{DG}}(x)\colon\, k = 0, \dots, n-1,\, x \in \Lambda_n\},$$
$$\{\xi_k^{\mathrm{GFF}}(x)\colon\, k = 0, \dots, n-1,\, x \in \Lambda_n\},$$

*and a family of independent zero-one valued random variables*

$$\{B_k(x)\colon\, x \in \Lambda_{n-k},\, k = 1, \dots, n-1\}$$

*such that the following holds:*

(1) $P\big(B_k(x) = 1\big) = \mathrm{e}^{-R_k}, \quad x \in \Lambda_{n-k},\, k = 1, \dots, n-1$

(2) *the families of Bernoulli random variables and GFF are independent of each other*

(3) $\forall\, k = 1, \dots, n-1$ *and* $\forall\, x \in \Lambda_n$,

$$\xi_k^{\mathrm{DG}}(x) = \xi_k^{\mathrm{GFF}}(x) \quad \text{on } \{B_k(m^k(x)) = 1\}$$

(4) $\forall\, k = 0, \dots, n-1$ *and* $\forall\, x \in \Lambda_n$,

$$P\Big(\big|\xi_k^{\mathrm{DG}}(x) - \xi_k^{\mathrm{GFF}}(x)\big| > C\Big) = 0$$

# Part II

# ① Tightness of the maximum of the DG-model

## Theorem (Tightness of DG-maximum)

$\forall\, \beta \in (0, \beta_c)$,

$$\lim_{u\to\infty} \sup_{n\geq 1} P_{n,\beta}\left(\Big|\max_{x\in\Lambda_n} \varphi_x^{\mathrm{DG}} - m_n\Big| \geq u\right) = 0$$

*and* $\forall\, \lambda > 0$,

$$\lim_{u\to\infty} \sup_{n\geq 1} P_{n,\beta}\left(\big|\{x \in \Lambda_n \colon \varphi_x^{\mathrm{DG}} \geq m_n - \lambda\}\big| \geq u\right) = 0$$

## Proposition

$\forall\, \beta \in (0, \beta_c)$, $\exists\, c', C' > 0$ *and,* $\forall\, t > 0$ *and* $u < t$, $\exists\, n_0 \geq 1$ *such that* $\forall\, n \geq n_0$ *and* $z \in \Lambda_n$,

$$P\left(\varphi_z^{\mathrm{DG}} \geq m_n + u, \max_{x\in\Lambda_n} \varphi_x^{\mathrm{GFF}} \leq m_n + t\right) \leq C' b^{-n}\big(1 + (t \vee (t - u + 1))^2 \beta\big) \mathrm{e}^{-\sqrt{\beta}\, c' u},$$

*where* $P$ *is the coupling law*

# ② Convergence of the extremal process of the DG-model

Let $f : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous with compact support. Define

$$Y_n = \sum_{x \in \Lambda_n} f\left([x]_n, \varphi_x^{\mathrm{DG}} - \lfloor m_n \rfloor\right)$$

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Fix $k > 0$. Given $z \in \Lambda_{n-k}$, $\forall\, x \in \Lambda_n$ with $m^k(x) = z$ (i.e. $x = (z, x_{k+1}, \cdots, x_n)$), abbreviate

$$\varphi_z^{\mathrm{DG}} := \sum_{j=k}^{n-1} \xi_j^{\mathrm{DG}}(x) \quad \text{and} \quad \varphi_z^{\mathrm{GFF}} := \sum_{j=k}^{n-1} \xi_j^{\mathrm{GFF}}(x)$$

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$$Y_n = \sum_{x \in \Lambda_n} f\left([x]_n, \varphi_x^{\mathrm{DG}} - \lfloor m_n \rfloor\right)$$

Fix $k > 0$. Given $z \in \Lambda_{n-k}$, $\forall\, x \in \Lambda_n$ with $m^k(x) = z$ (i.e. $x = (z, x_{k+1}, \cdots, x_n)$), abbreviate

$$\varphi_z^{\mathrm{DG}} := \sum_{j=k}^{n-1} \xi_j^{\mathrm{DG}}(x) \quad \text{and} \quad \varphi_z^{\mathrm{GFF}} := \sum_{j=k}^{n-1} \xi_j^{\mathrm{GFF}}(x)$$

Define $g_k : [0,1] \times \mathbb{R} \to [0,\infty)$ by

$$\mathrm{e}^{-g_k(v,h)} := E\left(\mathrm{e}^{-\sum_{x \in \Lambda_k(z)} f\left(v, \varphi_x^{\mathrm{DG}} - \varphi_z^{\mathrm{DG}} + h\right)} \,\middle|\, \varphi_z^{\mathrm{GFF}} = \varphi_z^{\mathrm{DG}} = h\right),$$

where $\Lambda_k(z) := \{x \in \Lambda_n \colon m^k(x) = z\}$

# ② Convergence of the extremal process of the DG-model

**Proposition (Conversion to GFF extremal process)**

$$\lim_{k\to\infty}\limsup_{n\to\infty}\left| E_{n,\beta}(\mathrm{e}^{-Y_n}) - E_{n-k,\beta}\left(\exp\left\{-\sum_{z\in\Lambda_{n-k}} g_k\left([z]_{n-k}, \varphi_z^{\mathrm{GFF}} - \lfloor m_n \rfloor\right)\right\}\right)\right| = 0$$

# ② Convergence of the extremal process of the DG-model

## Proposition (Conversion to GFF extremal process)

$$\lim_{k\to\infty} \limsup_{n\to\infty} \left| E_{n,\beta}(\mathrm{e}^{-Y_n}) - E_{n-k,\beta}\left(\exp\left\{-\sum_{z\in\Lambda_{n-k}} g_k\left([z]_{n-k}, \varphi_z^{\mathrm{GFF}} - \lfloor m_n \rfloor\right)\right\}\right)\right| = 0$$

- Thanks to the strong coupling, $\forall\, \epsilon > 0$

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\left(|Y_n - Y'_{n,k}| > \epsilon\right) = 0,$$

where

$$Y'_{n,k} := \sum_{x\in\Lambda_n} f\left([m^k(x)]_{n-k}, \varphi_x^{\mathrm{DG}} - \lfloor m_n \rfloor\right) 1_{\{B_j(x)=1 \forall j=k,\ldots,n-1\}}$$

# ② Convergence of the extremal process of the DG-model

Along increasing sequences $\{n_j\}_{j\in\mathbb{N}}$ of natural numbers for which $s := \lim_{j\to\infty}(m_{n_j} - \lfloor m_{n_j}\rfloor)$ exists,

$$E_{n-k,\beta}\left(\exp\Big\{-\sum_{z\in\Lambda_{n-k}} g_k\big([z]_{n-k}, \varphi_z^{\mathsf{GFF}} - \lfloor m_n\rfloor\big)\Big\}\right)$$

$$\underset{j\to\infty}{\longrightarrow} E\left(\exp\Big\{-\int Z(\mathrm{d}x)\otimes \mathrm{e}^{-\alpha h}\mathrm{d}h\otimes\nu(\mathrm{d}\chi)\big(1-\mathrm{e}^{-\int g_k(x,\, s+\beta^{-1/2}(h-\alpha k+\cdot))\,\mathrm{d}\chi}\big)\Big\}\right)$$

$$= E\left(\exp\Big\{-c\,\mathrm{e}^{\alpha\sqrt{\beta}s}\int Z(\mathrm{d}x)\otimes\nu_{\beta,k}(\mathrm{d}\chi)\sum_{n\in\mathbb{Z}}\mathrm{e}^{-\alpha\sqrt{\beta}\,n}\big(1-\mathrm{e}^{-\int f(x,n+\cdot)\mathrm{d}\chi}\big)\Big\}\right)$$

$$\underset{k\to\infty}{\longrightarrow} E\left(\exp\Big\{-c\,\mathrm{e}^{\alpha\sqrt{\beta}s}\int Z(\mathrm{d}x)\otimes\nu_{\beta}(\mathrm{d}\chi)\sum_{n\in\mathbb{Z}}\mathrm{e}^{-\alpha\sqrt{\beta}\,n}\big(1-\mathrm{e}^{-\int f(x,n+\cdot)\mathrm{d}\chi}\big)\Big\}\right)$$

# The end