Measure Theory, 620-645
Remarks on Homework Two

On the whole, the solutions were good, and I think an improvement on the first homework. With the first homework the actual grading, from my point of view, was bedeviled with many answers and choices of notation which strictly speaking did not appear to make mathematical sense. Those instances were much rarer this time around.

I do have one general comment though. It seemed that some people had a slight tendency to try to answer quite simple questions by quoting some big theorem. For instance, a couple of people looked up results on wikipedia to see if there was a theorem they could use.

The concern I have here is that it resembled a kind of overkill. None of the questions really required difficult techniques, beyond what we were doing every week in the classes anyway.[1] Using a big theorem, or a refined statement of Lusin's theorem which you can find in another text, to solve a straightforward problem that only required elementary techniques, is perhaps not the best way to try to really understand the subject matter at hand.

That said, here are a few selected remarks.

Q2: *Let Σ be a σ-algebra on a set X and let μ be a σ-finite measure on Σ and assume $\mu(X) = \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from X to $\mathbb{R}$.*

(i) Show that the set of x for which $(f_n(x))_{n\in\mathbb{N}}$ converges is measurable.

For instance, you can observe that at each n, m, k, the set $B_{n,m,l}$,

$$\{x \in X : |f_n(x) - f_m(x)| < \frac{1}{k}\},$$

will be in Σ by the assumptions on f_n and f_m. (Strictly speaking you should probably prove this, but it is straightforward). Then the set of points x for which the sequence of values converges equals

$$\bigcap_k \bigcup_N \bigcap_{n,m>N} B_{n,m,l},$$

and hence is in Σ. A similar kind of argument will also solve (ii).

(iii) Show under these assumptions, of pointwise convergence everywhere, there exists for any real $c > 0$ some $A_c \subset X$ in Σ with $\mu(A_c) > c$ and the f_n's converging uniformly on A_c.

This can be reduced to Egorov's theorem by going to a subset of X with finite measure greater than c. The main mistake which did occur is when people concluded that if

$$X = \bigcup A_m$$

and the (f_n) sequence is uniformly convergent on each A_m then it must be uniformly convergent on X. This is a kind of logical slip up. For instance, if we consider $f_n : \mathbb{R} \to \mathbb{R}$ with $f_n(x) = \min\{x^2, n\}$ and $A_m = [-m, m]$, then the f_n's converge uniformly on each A_n to f, but on $\mathbb{R} = \bigcup_n A_n$ they do not.

Q5: *Let $f : [0, 1] \to \mathbb{R}$ be integrable (with respect to Lebesgue measure). Show that for any $\epsilon > 0$ there is a continuous function $g : [0, 1] \to \mathbb{R}$ such that*

$$\int_{[0,1]} |f - g| d\mu < \epsilon.$$

To my embarrassment, I noticed when reading the solutions that this was basically answered by 4.12 from the course notes. There is another way to prove this, which actually has the stronger conclusion that if μ is any Borel probability

[1] At the risk of repeating myself, I really do think that the best way to master a course like this one is to go home every day after the lectures and make sure you understand every step of the proofs given that day.

measure on the unit interval, and $f : [0,1] \to \mathbb{R}$ be integrable with respect to μ, then for any $\epsilon > 0$ there is a continuous function $g : [0,1] \to \mathbb{R}$ such that

$$\int_{[0,1]} |f - g| d\mu < \epsilon.$$

One starts off with Lusin's theorem from the course, which gives us a compact set K with measure very close to 1. Using the kinds of exhaustion arguments we saw in the section on integration (e.g. lemma 4.4) we can actually ensure that $\int_{K^c} |f| d\mu < \epsilon/2$. Then $f|_K$ is continuous and K is compact, we can find $a > 0$ such that $f(x) \in [-a, a]$ all $x \in K$.

Fix $C \subset [0,1] \setminus K$ which is closed and for which $\mu([0,1] \setminus (C \cup K))$ is less than $\epsilon/(4a)$. Now using some basic topology (the hint attached to the footnote of this problem) we can obtain a continuous function

$$g : [0,1] \to [-a, a]$$

which extends f and has the property that $g(x) = 0$ all $x \in C$. It is then a routine calculation to see g is as required.

Q6: *Let X be a Polish space and let μ be a Borel probability measure on X. Let $f : X \to \mathbb{R}$ be measurable (with respect to μ) – which is to say that $f^{-1}[V]$ is measurable (with respect to μ) for any open $V \subset \mathbb{R}$.*

Show that there is a Borel function $g : X \to \mathbb{R}$ and a conull set $B \subset X$ such that f and g agree on B.

Okay. So this was definitely a problem where a lot of people resorted to unnecessary complications.

Let $\{V_n : n \in \mathbb{N}\}$ be a countable basis for $\mathbb{R}$. At each n we have that $f^{-1}[V_n]$ is measurable, hence we can find Borel B_n, C_n with

$$B_n \subset f^{-1}[V_n] \subset C_n,$$
$$\mu(C_n \setminus B_n) = 0.$$

Now let N be the union

$$\bigcup_{n \in \mathbb{N}} C_n \setminus B_n.$$

N is null and $f|_{N^c}$ is Borel, since the pullback of each V_n equals $N^c \cap B_n$.

(Please note, the ideas used here are very close to ideas which appeared in the proof of Lusin's theorem among other places.)

Q7: *(i) Show that if (X, Σ, μ) is a standard Borel probability space and $(B_n)_{n \in \mathbb{N}}$ is a sequence of sets in Σ, then $\mu(\bigcap_{n \in \mathbb{N}} B_n) = \lim_{N \to \infty} \mu(\bigcap_{n \leq N} B_n)$.*

(ii) Show that (i) above might fail if we simply assume μ to be a σ-finite measure on a standard Borel space (X, Σ).

This was intended to be an easy problem. The argument we gave in the course for lemma 6.1 exactly works to solve Q7(i) – and in fact, properly understood 6.1 has Q7(i) as a corollary since all finite measures are in particular signed measures. As for a counterexample to Q7(ii), I presented one explicitly in class around the time we prove lemma 6.1.