The Surface of Revolution Case Worked Out

Let $(x(s), y(s))$ be an arclength parameter plane curve with $y(s)>0$ for all s. Let $s=u$, $\theta=v$ & $S(s,\theta) = S(u,v) = (x(u), y(u)\cos v, y(u)\sin v)$

Then $S_u = (x', y'\cos v, y'\sin v)$ where $x' = \frac{dx}{du}$

$S_v = (0, -y\sin v, y\cos v)$ $\quad y' = \frac{dy}{du}$

So $S_u \times S_v = (yy', -x'y\cos v, -x'y\sin v)$.

Check $S_u \times S_v \cdot S_v = y^2 x' \sin v\cos v - x'y^2\cos v\sin v = 0$ ✓

$S_u \times S_v \cdot S_u = x'y'y - x'y'\cos^2 v - x'y'y\sin^2 v = 0$ ✓

The unit normal is $(S_u \times S_v)/\|S_u \times S_v\|$

$\|S_u \times S_v\|^2 = y^2(y')^2 + (x')^2 y^2\cos^2 v + (x')^2 y^2 \sin^2 v$

$= (y)^2((y')^2 + (x')^2) = y^2$

So $\|S_u \times S_v\| = y$ and $N = (y', -x'\cos v, -x'\sin v)$.

Check: When $v=0$, this should be ($\pm$) the curve normal of the plane curve that is being rotated and indeed $(y', -x')$ is $\perp$ (x', y') and unit length.

Next: $E = \langle S_u, S_u\rangle = (x')^2 + (y')^2(\cos^2 v + \sin^2 v) = 1$

$F = \langle S_u, S_v\rangle = yy'(-\cos v\sin v + \sin v\cos v) = 0$

$G = y^2\sin^2 v + y^2\cos^2 v = y^2$.

$S_{uu} = (x'', y''\cos v, y''\sin v)$. So

$L_{11} = -\langle N, S_{uu}\rangle = -(x''y' - x'y''\cos^2 v - x'y''\sin^2 v)$

$= -(x''y' - x'y'')$

$L_{12} = -\langle N, S_{uv}\rangle$ and $S_{uv} = +(0, -y'\sin v, y'\cos v)$

so $L_{12} = -(x'y'\cos v\sin v - x'y'\cos v\sin v) = 0$

Finally $L_{22} = -\langle N, S_{vv}\rangle$ & $S_{vv} = (0, -y\cos v, -y\sin v)$

So $L_{22} = -x'y(\cos^2 v + \sin^2 v) = -x'y$

Gauss curvature $= (x''y' - x'y'')(x'y)/y^2 = (x''x'y' - (x')^2 y'')/y$

Recall $(x')^2 + (y')^2 = 1$. ②

Now $x''x' + y''y' = \frac{1}{2}((x')^2+(y')^2)' = 0$ &

$x''x' = -y''y'$

So $(x''y' - x'y'')x'/y$

$= (x''x'y' - (x')^2 y'')/y$

$= \left(\frac{1}{y}\right)\left[-(y''y')(y') - (x')^2 y''\right] = \dfrac{-y''((y')^2+(x')^2)}{y}$

$= -y''/y$.

Thus

Gauss curvature $= -y''/y$.

We illustrate with the unit sphere: $s \in (0, \pi)$

$(x(s), y(s)) = (-\cos s, \sin s)$ (we like to think of $x(s)$ as an increasing function of s)

Then Gauss curvature $= -y''/y$

$= -(-\sin s)/\sin s = +1$

as we calculated before (e.g. from standard "graphing over the tangent plane" form of the sphere).

Of course this would have worked in the unsimplified $x'(x''y' - x'y'')/y$ form. This $=$ (just to check)

$(\sin s)(\cos s \cdot \cos s - (+\sin s)(\sin s))/\sin s = +1$.

Naturally, this had to work out since $(x(s), y(s))$ is indeed arclength parameter here.