Summary of Surface Theory (for Midterm II)

Definition of (nonsingular) "surface patch"
$S: U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$ open, $S(u,v)$, S one-to-one,
$S_u \times S_v \neq \vec{0}$ (S is differentiable)
Unit normal $N = S_u \times S_v / (\| S_u \times S_v \|)$.
Tangent plane: linear span of S_u and $S_v = \{aS_u + bS_v\}$
(= orthogonal complement of N).
First fundamental form: "quadratic form" on tangent plane sending $aS_u + bS_v$ to
$Ea^2 + 2Fab + Gb^2$ where $E = \langle S_u, S_u \rangle$, $F = \langle S_u, S_v \rangle$
and $G = \langle S_v, S_v \rangle$. This is $\| aS_u + bS_v \|^2$.
Application: arclength of $S(u(t), v(t))$

$$= \int \left[E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\cdot\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2 \right]^{1/2} dt$$

"Second fundamental form": $L_{11} = \langle N_u, S_u \rangle \left(= -\langle N, S_{uu} \rangle\right)$
$L_{12} = \langle N_u, S_v \rangle = -\langle N, S_{uv} \rangle$
$L_{22} = \langle N_v, S_v \rangle = -\langle N, S_{vv} \rangle$.
[This also gives a quadratic form: $aS_u + bS_v$ goes to
$L_{11}a^2 + 2L_{12}ab + L_{22}b^2$]

Basic example: $S(u,v) = (u, v, f(u,v))$.
$S_u = (1, 0, f_u)$ $S_v = (0, 1, f_v)$. <u>Assume</u> $f_u = f_v = 0$ at $(0,0)$
Then N at $(0,0) = (0,0,1)$.
$E = 1 + (f_u)^2$, which $= 1$ at $(0,0)$
$F = f_u f_v$, which $= 0$ at $(0,0)$
$G = 1 + (f_v)^2$, which $= 1$ at $(0,0)$

$S_{uu} = (0, 0, f_{uu})$, $S_{uv} = (0, 0, f_{uv})$, $S_{vv} = (0, 0, f_{vv})$

At $(0,0)$, since $N = (0,0,1)$

$L_{11} = -f_{uu} \quad L_{12} = -f_{uv} \quad L_{22} = -f_{vv}$.

! [Definition of Gauss curvature

$$K = (L_{11}L_{22} - L_{12}^2)/(EG - F^2)$$

So in example, $K = f_{uu}f_{vv} - f_{uv}^2$ (familiar item from two-variable calculus (max/min test, etc.)

Gauss curvature is "invariant": independent of reparameterization [idea of reparameterization:

$S: U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$, $(u,v) \in U$, $F: V \to U$ $\hat{u}, \hat{v} \in V \subset \mathbb{R}^2$

get new surface $\hat{S}(\hat{u}, \hat{v}) = S(F(\hat{u}, \hat{v}))$

Geometrically, the same surface. Assume here

$$\det\begin{pmatrix} \frac{\partial u}{\partial \hat{u}} & \frac{\partial u}{\partial \hat{v}} \\ \frac{\partial v}{\partial \hat{u}} & \frac{\partial v}{\partial \hat{v}} \end{pmatrix} \neq 0,$$ then $\hat{S}$ nonsingular. Then, e.g.,

$$\hat{E} = \langle \hat{S}_{\hat{u}}, \hat{S}_{\hat{u}} \rangle \text{ \& } \hat{S}_{\hat{u}} = \frac{\partial u}{\partial \hat{u}} S_u + \frac{\partial v}{\partial \hat{u}} S_v$$

So $$\hat{E} = \left(\frac{\partial u}{\partial \hat{u}}\right)^2 E + 2\frac{\partial u}{\partial \hat{u}}\frac{\partial v}{\partial \hat{u}} F + \left(\frac{\partial v}{\partial \hat{u}}\right)^2 G$$ etc.

same for L's.

Calculation shows Gauss curvature of $\hat{S}$ and S are the same: Gauss curvature is "geometric"]

Big Theorem (Gauss "Theorema Egregium"):

Gauss curvature is intrinsic, i.e., determined by E, F, G and their derivatives. [Even though L_{11}, L_{12}, and L_{22} which are not determined by E, F and G and derivatives of them, occur in definition of Gauss curvature]

Know proof!

Gauss curvature = expression computable from E, F, and G and derivatives

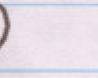
Other important example (along with $(u, v, f(u,v))$)

Surface of revolution $S(u,v) = (x(u), y(u)\cos v, y(u)\sin v)$

where (wolog) $(x')^2 + (y')^2 = 1$.

Gauss curvature $= -y''/y$ $\quad$ ($'$ = u-derivative)

Gauss-Bonnet Theorem: $\int$ Gauss curvature $d(\text{area})$

$= 4\pi$, (if surface is a topological sphere 

$= 0$ $\quad$ if surface is a torus 

$= -4\pi$ $\quad$ if surface has "two holes" 

Verification of spherical case for surface of revolution

Theorem: If S is a bounded closed (no "edges") surface in $\mathbb{R}^3$, then there is a point of S where the Gauss curvature is positive.

Proof: Look at $p \in S$ when $\|S(u,v)\|^2$ has its maximum value. S touches a sphere from the inside there. Then apply:

Important: not just for this one proof! →

[Quadratic form observations (handout) on eigenvalues being max, min values on unit circle, etc.

(You need this to see $f \geq g \geq 0$ $\quad$ $f_u = f_v = 0$ at $(0,0)$

$g_u = g_v = 0$ at $(0,0)$, $f(0,0) = g(0,0) = 0 \Rightarrow$

$f_{uu} f_{vv} - f_{uv}^2 \geq g_{uu} g_{vv} - g_{uv}^2$ at $(0,0)$ to complete proof]

Behavior of Gauss curvature under scaling: λS has Gauss curvature $(1/\lambda^2)$ Gauss curvature of S.