The Isoperimetric Inequality

Theorem: If $\sigma(s)$, $s \in [0, 2\pi]$ is a smoothly closed (regular) simple curve in $\mathbb{R}^2$ with arclength parameter (and hence length $= 2\pi$), then the area of its interior is $\leq \pi$ with equality if and only if it is a circle of radius 1.

Proof (Hurwitz): By translation if necessary, we can and shall assume that, with $\sigma(s) = (x(s), y(s))$, (we assume curve traversed counterclockwise) $\int_0^{2\pi} y(s)\, ds = 0$. By Green's Theorem, the area A of the interior is $\int_0^{2\pi} -y(s) \frac{dx}{ds}\, ds$. So

$$\pi - A = \frac{1}{2}\int_0^{2\pi} \left(1 + 2y(s)\frac{dx}{ds}\right) ds$$

(switching to $'$ notation: $x' = \frac{dx}{ds}$ etc.)

$$= \frac{1}{2}\int (x')^2 + (y')^2 + 2yx' = \frac{1}{2}\int_0^{2\pi} (x'+y)^2 ds + \frac{1}{2}\int_0^{2\pi} [(y')^2 - y^2]\, ds$$

The first integral is ≥ 0 since its integrand is nonnegative. Now $y(s)$ is a periodic function with period 2π (smooth at the endpoints) and with $\int_0^{2\pi} y = 0$. So such functions it is known (from Fourier series) that $\int_0^{2\pi} (y')^2 \geq \int_0^{2\pi} y^2$ with equality if and only if y is a linear combination of cos and sin. Thus $\pi - A \geq 0$ and $\pi = A$ if and only if $y(s) = a \cos s + b \sin s$ ($a^2 + b^2 \leq 1$ since for all s $(y')^2 \leq 1$) while $x' = -y$. Since x' vanishes for some s_0 (otherwise the curve could not be closed), $a^2 + b^2 = 1$ ($y'(s_0) = -a \sin s_0 + b \cos s_0$ which can have $|\ | = 1$ only if $a^2 + b^2 = 1$, given that $a^2 + b^2 \leq 1$).

By reparametrization (translating s) $y = \sin s$. Then

$$x(s) = \int_0^s -\sin u\, du = c + \cos s.$$

The curve is a unit circle with center $(c, 0)$. $\square$