Isoperimetric Inequality and So On:

Deformation Calculations: $\left.\frac{dA_t}{dt}\right|_0$ & $\left.\frac{dL_t}{dt}\right|_0$

Set-up: $\gamma_0 : [0, L] \to \mathbb{R}^2$ arclength parameter curve, smoothly closed, simple

$$\vec{\gamma}_t(s) = \vec{\gamma}_0(s) + f_t(s)\,T(s) + g_t(s)\,N(s)$$

where $T(s)$ = tangent vector of γ_0, $N(s)$ = normal vector of γ_0 and $f_t(s)$, $g_t(s)$ satisfy $f_0(s) = 0$ $g_0(s) = 0$ all s (and f_t, g_t are smoothly periodic so that γ_t is smoothly closed (in s) for each t).

<u>Note:</u> $\gamma_t(s)$ may not have s as arc-length parameter.

Notation to make calculation look nice

$$a(s) = \left.\frac{\partial f_t(s)}{\partial t}\right|_{t=0} \qquad b(s) = \left.\frac{\partial g_t(s)}{\partial t}\right|_{t=0}$$

so that
$$f_t(s) = 0 + t\,a(s) + \begin{pmatrix}\text{terms of order}\\ \text{2 or higher}\\ \text{in } t\end{pmatrix}$$
$$g_t(s) = 0 + t\,b(s) + \begin{pmatrix}\text{terms of order 2}\\ \text{or higher in } t\end{pmatrix}$$

In computing $\left.\frac{dA_t}{dt}\right|_{t=0}$ and $\left.\frac{dL_t}{dt}\right|_{t=0}$ we

can (and shall) ignore the terms of order 2 or higher in t. Hereafter we do this without saying so, sometimes anyway.

We write $'$ for s derivative.

Now
$$\frac{d\gamma_t(s)}{ds} = \gamma_t'(s)$$
$$= \gamma_0' + (taT)' + (tbN)'$$
(remember we are ignoring terms of order 2 or more in t)
$$= T(s) + t\,a'T + (ta)T' + tb'N + tbN'$$
$$= (1 + ta' - tbK)T + (tb' + taK)N$$
where we have used $T' = kN$ and $N' = -KT$.

So (up to order 2 in t terms)
$$\|\gamma_t'(s)\| = \sqrt{(1 + t(a' - bK))^2 + (tb' + taK)^2}$$
$$= \sqrt{1 + 2t(a' - bK) + \text{higher order in } t}$$
$$= 1 + t(a' - bK) + \text{h.order}$$

So
$$L_t = \int_0^L \left(1 + t(a' - bK) + \left(\underset{\text{in } t}{\text{h.order}}\right)\right) ds$$
and hence
$$\left.\frac{dL_t}{dt}\right|_{t=0} = \int a' - bK \; ds$$
$$= -\int bK \; ds$$
because $\int a' = 0$ by (smooth) periodicity.

Summary
$$\left.\frac{dL_t}{dt}\right|_{t=0} = -\int_0^L K(s)\,b(s)\,ds$$

Now we want to do the same thing for A_t, namely to compute $\frac{dA_t}{dt}\Big|_{t=0}$. For this,

we note that (by Green's Theorem), the area inside a curve $(x(s), y(s))$ (simple closed curve) is

$$\frac{1}{2}\int -y\frac{dx}{ds} + x\frac{dy}{ds} = \frac{1}{2}\int \gamma \times \gamma'$$

(where $(\alpha,\beta)\times(c,d) = \alpha d - \beta c$ as usual for vectors in the plane). This works whether or not s is arclength parameter. In particular, we apply this to (ignoring higher order than 1 in t)

$$\gamma_t(s) = \gamma_0(s) + t\,a(s)\,T + t\,b(s)\,N$$

so

$$\gamma_t'(s) = \gamma_0'(s) + t\,(aT)' + t(bN)'$$
$$= T + t(aT)' + t(bN)'$$

Now with $\gamma = \gamma_0$:

$$2A_t = \int \gamma_t \times \gamma_t' = \int \begin{matrix} \gamma\times\gamma' + t\,\gamma\times(aT)' \\ + t\,\gamma\times(bN)' + t\,b\,N\times T \end{matrix}$$

(throwing out t^2 terms). The $\gamma' \times t\,a(s)T$ term $=0$ since $\gamma' = T$!! as shown in second $\gamma_t'(s)$ formula

Note also that $\int \gamma\times(aT)' = -\int \gamma'\times aT = -\int T\times aT = 0$

by integration by parts (no end terms by periodicity) while

$$\int \gamma\times(bN)' = -\int \gamma'\times bN = -\int b\,T\times N$$
$$= -\int b$$

and $\int b\,N\times T = -\int b$ also.

Combining, we get

$$2A_t = \int \gamma \times \gamma' \doteq 2t\int b \,, \quad \text{and} \quad \int \gamma \times \gamma' = 2A_0$$

So $A_t = A_0 - t\int b$ (+ higher order than 1 terms in t)

Hence $\dfrac{dA_t}{dt} = -\int_0^L b(s)\,ds.$

Important Observation: If $b(s)$ is a function with $\int_0^L b(s)\,ds = 0$, then there is a constant-area deformation $\gamma_t(s)$, i.e., A_t is constant, with (given γ_0 as before)

$$\gamma_t(s) = \gamma_0(s) + t\,b(s)\,N(s)$$
$$+ \text{(higher order than 1 in } t \text{ terms)}$$

Proof: Let $\hat{\gamma}_t(s) = \gamma_0(s) + t\,b(s)\,N(s)$. The corresponding area function $\hat{A}_t$ may not be constant but it does have

$\left.\dfrac{d\hat{A}_t}{dt}\right|_{t=0} = 0$ (by the calculation just given).

Set $\gamma_t(s) = \sqrt{\dfrac{A_0}{\hat{A}_t}}\,\hat{\gamma}_t(s)$, so the area inside γ_t is constant.

Since $\hat{A}_t = A_0 + 0t + \left(\text{term of order at least 2 in } t\right)$,

it follows that $\gamma_t(s) - \hat{\gamma}_t(s)$ is order at least 2 in t so

$$\gamma_t(s) = \gamma_0(s) + t\,b(s)\,N(s) + \left(\begin{matrix}\text{term} \\ \text{of order} \\ \text{at least} \\ 2 \text{ in } t\end{matrix}\right)$$

as required.

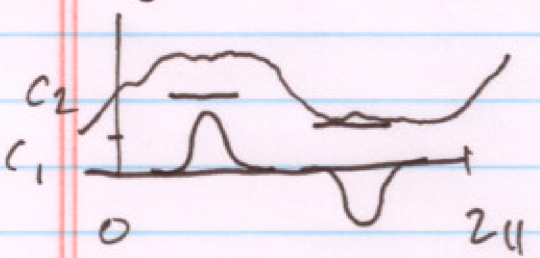
With these considerations in mind, we can ~~know~~ now prove our main result:

Theorem: If $\gamma : [0, 2\pi] \to \mathbb{R}^2$ is a simple, smoothly closed, arclength-parameter curve with the property that the area it encloses ~~is maximum~~ is $\geq$ the area enclosed by any other curve of length 2π, then γ is a unit circle (of area π)

Proof: By a straightforward rescaling argument, maximizing area for given length is the same as minimizing length for given area. So under the conditions given, if ~~γ_t~~ $\gamma_t(s)$, $s \in [0, 2\pi]$, is a deformation with A_t constant, $\gamma_0(s) = \gamma(s)$, hence A_0 = area inside γ, then $L_t \geq L_0$ = length of $\gamma = 2\pi$ (If not, rescaling an γ_t with $L_t < L_0 = 2\pi$ would give a curve $\alpha \gamma_t$, $\alpha > 1$, $\alpha \in \mathbb{R}$, of length 2π and area $= \alpha^2 A_0 > A_0$).

Suppose γ does not have constant curvature, that is, $k(s)$ is not constant. Then there is a function $b(s)$ (2π periodic) with $\int_0^{2\pi} b(s)\, ds = 0$ but $\int_0^{2\pi} k(s) b(s) \neq 0$.

(This is an easy argument: take two same-size intervals with $k \geq c_1$ ~~on~~ on one, $k \leq c_1$, $c_1 < c_2$, on the both, make b positive on one, negative on the other, symmetrically as shown



$\int b = 0$, $\int \kappa b \neq 0$.)

By the observation on page 4, there is a "variation" $\gamma_t(s)$ of the form
$\gamma_t(s) = \gamma_0(s) + t\, b(s)\, N(s) + (\text{h.o. in } t)$
that has A_t constant, i.e., always $= A_0$.
Then
$$\left.\frac{dL_t}{dt}\right|_{t=0} = -\int \kappa b \neq 0 .$$

So, for t small in absolute value, positive if $\int \kappa b > 0$, negative if $\int \kappa b < 0$, we have that $L_t < L_0$, a contradiction.

Thus κ is constant for a curve γ with minimal length enclosing a given area (or, equivalently, maximal area for a given length). Hence such a γ is a circle. In particular, if the length is 2π, it is a unit circle. □