Lecture 5 --Part I
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One of the crucial differences between Kähler and non- Kähler geometry has to do with the difference between a closed form of type (p,p) being exact in the usual sense of being in the image of d versus the same form being expressible as $\partial \bar{\partial}$ of another form. On Kähler manifolds these two things are the same. This is the fact known as the $\partial \bar{\partial}$ Lemma, derived from Hodge Theory for Kähler manifolds. [ see the insert for details]. But on non- Kähler manifolds they are potentially different.

**[***Insert*: **The** $\partial\bar{\partial}$ **Lemma**: On a compact Kähler manifold, if a form $\omega$ of type (p,p) is d-exact, then there is a (p-1,p-1) form $\theta$ such that $\partial\bar{\partial}\theta=\omega$ .
The proof involves harmonic theory. For this, recall that if D is any one of the operators d, $\partial$ or $\bar{\partial}$, then there is a harmonic decomposition :

$$\text{image (D)} \;+\; \text{image }(D^*) \;+\; \ker (DD^* + D^*D)$$

where $D^*$ is the formal adjoint of D relative to the inner product on forms determined by the Kähler metric. This decomposition is orthogonal relative to the (integrated) inner product of forms. The crucial fact for our purposes is that (because the metric is Kähler), the operator $DD^* + D^*D$, the D-Laplacian, is the SAME operator, up to constant factors, whichever one of the D possibilies is involved. The ordinary, real Laplacian extended to act on complex forms by complex linearity preserves (p,q) type and is the same operator , except for a factor of ¼, as the $\bar{\partial}$ or $\partial$ Laplacians . So a (p,q) form is harmonic (Laplacian =0) in the d-sense if and only if it is $\bar{\partial}$ harmonic if and only if it is $\partial$-harmonic.

Turning now to the situation of an exact form of type (p,p), write $\omega = \mathrm{d}(\alpha+\beta)$ where $\alpha$ is type (p,p-1) and $\beta$ is type (p-1, p). Since $d = \partial + \bar{\partial}$ we must have

$$\begin{aligned}\omega &= (\partial+\bar{\partial})(\alpha+\beta)\\ &= \partial\alpha + \bar{\partial}\alpha + \bar{\partial}\beta + \partial\beta\end{aligned}$$

Examining types, we get $\partial\alpha = 0$ and $\bar{\partial}\beta = 0$ so that $\omega = \bar{\partial}\alpha + \partial\beta$ .
Now note that since $\partial\alpha = 0$ , the Hodge decomposition for $\partial$ from above gives that $\alpha = \partial f + h$ , where f is a (p-1, p-1) form and h is a (p,p-1) form that is $\partial$-harmonic. Now note that being $\partial$ harmonic is the same thing as being $\bar{\partial}$-harmonic so that h is $\bar{\partial}$-harmonic and in particular $\bar{\partial}$ h=0. Thus $\bar{\partial}\alpha = \bar{\partial}\partial f$.
Similarly there is a (p-1,p-1) form g such that $\partial\beta = \partial\bar{\partial} g$.
Since $\bar{\partial}\partial = -\partial\bar{\partial}$ , we obtain that $\omega = \partial\bar{\partial}(-f+g)$. *End of insert***]**

For example, on $S^6$, if it had a complex structure, the first Chern form would be exact, because the deRham cohomology $H^2(S^6, R)$ is 0. But in principle it could fail to be the case that the (1,1) form representing the first Chern form is expressible as $\partial\bar{\partial}$ of a function. (Note that a complex structure on $S^6$ , if there is one , necessarily cannot admit a Kähler metric, because of the vanishing of the 2-cohomology of $S^6$). In fact, just for fun and to illustrate the importance of the $\partial\bar{\partial}$ Lemma, we give a proof that there is no complex structure on $S^6$ for which the first Chern form is expressible as $\partial\bar{\partial}$ of a function (for some Hermitian metric on $S^6$: note that the first Chern form is uniquely determined up to $\partial\bar{\partial}$ of a function in any case):

To prove this, suppose that $S^6$ did have such a complex structure, and fix a Hermitian metric. Then let $(z_1, z_2, z_3)$ be a holomorphic local coordinate system. For the fixed Hermitian metric, the volume form can of course be written locally as $V dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$ for some positive function V on the coordinate patch. The first Chern form is then given by $c_1 = -\frac{i}{2\pi}\partial\bar{\partial}\log V$.
As usual, the formula is local but the differential form defined is global. [easy and standard calculation: changing coordinates changes V by the real Jacobian of the coordinate transformation, but this is the absolute value squared of the holomporphic Jacobian J. Since $\log|J|^2$ is locally expressible as the sum of a holomorphic and a conjugate holomorphic function, it is annihilated by $\partial\bar{\partial}$ .]

The form $c_1$ is closed and hence, by vanishing of the second deRham cohomology, is expressible as d of some global 1 form. Suppose that in fact $c_1$ were $\partial\bar{\partial}$ f, for some function f. Then it would follow that $\partial\bar{\partial}\log(e^f V) = 0$, for any locally defined V defined as earlier. That is , $\log e^f V$ is plurisubharmonic, and hence is the real part of some holomorphic function h (locally, one can assume this works by shrinking coordinate systems if necessary). Thus $e^f V$ itself is expressed $|\exp(h/2)|^2$ . So labeling our coordinate cover by index α, we have a collection of holomorphic functions $F_\alpha$ with the holomorphic (3,0) forms $\Omega_\alpha$ defined by $F_\alpha dz_1 \wedge dz_2 \wedge dz_3$ having the property that $e^f \cdot (\text{volume form}) = \Omega_\alpha \wedge \bar{\Omega}_\alpha$ ., namely $F_\alpha = \exp(h/2)$ for the corresponding h.

Note that in this situation, one obtains that the quotient $F_\alpha / F_\beta$ has absolute value 1. Since it is holomorphic, it follows that it is constant, so that there exist constants $\theta_{\alpha\beta}$ with $F_\alpha/F_\beta = \exp(2\pi i\, \theta_{\alpha\beta})$ . These θ 's form a Cech cocycle with coefficients in R/Z. From the topology of $S^6$ (that the first cohomology is 0 for any coefficients) , one deduces that there exist $\theta_\alpha$ ,constants in R/Z, with $\theta_{\alpha\beta} = \theta_\alpha - \theta_\beta$. Then the form $\exp(-2\pi i\,\theta_\alpha)\,\Omega_\alpha$ is globally defined. It is clearly closed: $\partial = 0$ by type considerations and $\bar{\partial} = 0$ because it is holomorphic. It cannot be exact since the integral of it wedged with its conjugate is positive. Thus it represents a nontrivial element in the deRham cohomlogy $H^3(S^6, R)$ , a contradiction.

Note that this most definitely does not prove that $S^6$ has no complex structure!! Indeed, it conjecturally does have one. Rather, it proves any complex structure on $S^6$, which of course would clearly be non- Kähler, would also have to fail to satisfy the $\partial\bar{\partial}$ Lemma.

**Hermitian Yang Mills—continued** (Li, Yau, Zheng revisited)

We return to considering not general vector bundles, but specifically the tangent bundle in the Inoue case. The tangent bundle is stable (this follows from the Inoue condition contradicted: recall that the idea is to prove that there are no manifolds in the class with the property that for all line bundles the space of holomorphic sections of the holomorphic cotangent bundle tensored with the line bundle is 0 alone). Stability implies that there is an Hermitian Yang-Mills connection, that is a metric h such that the curvature form of the associated type (1,0) Hermitian connection has curvature form $F_h$ of the type $\alpha \otimes \mathrm{Id}$. This is called a "projectively flat connection". (Note that the statement that $\alpha=0$ in this case in the first Li, Yau, Zheng paper is in error, corrected in their second paper [Illinois Journal]. ) To complete the proof, we need to classify the complex surfaces(or more generally complex manifolds of all dimensions)for which such a connection exists. The exact result is this:

**Theorem**: Suppose that M is an Hermitian manifold with Kähler form $\omega$ satisfying $\partial\bar{\partial}\omega^{n-1} = 0$ and with the tangent bundle projectively flat. Then either

(1) M is flat, that is $F_h=0$, and balanced in the sense that $d\omega^{n-1}=0$

or

(2) M has a finite cover by a Hopf manifold, that is a manifold of the form $C^n - \{0\}$ / a Z action.

Note: This solves the Inoue question since ,with n=2, the first case would make the manifold Kähler, contradicting $b_1= 1$, while the manifold is not finitely covered by a Hopf manifold in the Inoue situation.

Proof of the Theorem

(We want to look at this because it illustrates a method of getting geometric structures. Later on we shall see that this is in fact an instance of a typical way to build geometric structures. In particular, one can do similar things for Hermitian symmetric spaces. If only the n=2 specific case here were of interest, one could in fact do the proof more simply, but the general method is of interest.)

The proof will be completed in Part II.