An Interlude on Curvature and Hermitian Yang Mills

As always, the story begins with Riemann surfaces or just (real) surfaces. (As we have already noted, these are nearly the same thing).

Suppose we wanted somehow to find a metric on S^2, conformal to a given metric, that had constant curvature. Actually, that such a metric exists is guaranteed by the Uniformization Theorem as follows: Given any metric, there is a conformal (Riemann surface) structure, as we have already noted. By Uniformization, this structure is biholomorphic to the standard CP^1 structure. So the pullback of the standard (round) Riemannian metric on S^2, which is associated to the usual complex structure on CP^1, solves our problem!

But suppose we did not know about uniformization (we are not going to have it in higher dimensions!). How would we find the conformal constant curvature metric? (Finding it would of course PROVE the uniformization in this case.) What we need is a characterization of the constant curvature metric among all (conformal to the given metric) metrics. Here is how to do it:

By the Cauchy Schwarz inequality, $\left(\int K^2 d(\text{area})\right)^{\frac{1}{2}} \cdot \left(\int 1 d(\text{area})\right)^{\frac{1}{2}} \geq \int |K| d(\text{area})$.

But $\int K d(\text{area}) = 4\pi$ by the Gauss Bonnet Theorem.

So $\int K^2 d(\text{area}) \geq \dfrac{(4\pi)^2}{\text{area}}$ where area is in the given new, conformal metric. If we assume that we have scaled by a constant so that the area is 4π, then we have

$\int K^2 d(\text{area}) \geq 4\pi$ with equality if and only if K is (a constant multiple of) 1. Actually, for this particular choice of area normalization, K is the constant 1, but K would have to be some constant, whichever area normalization we chose.

Thus we have reduced finding a conformal constant curvature metric to a minimization problem.

Similar considerations can be used to characterize the constant curvature metrics on other compact Riemann surfaces.

This suggests that one really ought to look for "canonical metrics" that minimize curvature integrals of one sort or another. This idea can be carried quite far, as we shall see.

It is, however, not entirely reasonable to expect to carry it out on compact Riemannian manifolds as a whole. Indeed, there are results of Nabutovsky and Weinberg that show that this cannot really be expected at all in complete generality.

What is reasonable to try is the following problem:

Let M be a compact Kähler manifold with Kähler form ω. Then try to minimize the curvature square integral on M among all Kähler metrics with the same Kähler class as ω, that is with Kähler form that is in the cohomology class of ω. This is in fact exactly what we would be doing on compact Riemann surfaces if we looked for mimizing the integral of K^2 within a conformal class of metrics with fixed area. The conformal class being fixed means we have a fixed complex structure and the family of Kähler metrics relative to it, and fixing the area corresponds precisely to fixing the cohomology class of the Kähler form(since $H^2(M,R)$ is mapped isomorphically to R by evaluating classes on the oriented fundamental cycle for M , and this is equivalent for forms to integrating the Kähler form over M, which of course gives the area of M relative to the Kähler metric)

Of course, the question arises: which curvature should we try to minimize the square integral of? As it happens, it does not matter! Calabi ("Extremal Kähler Metrics" in "Seminar on Differential Geometry" , Yau(ed.) , Princeton University Press, 1982) showed that the differences among the integral of the square of the scalar curvature, (one half of) the integral of the norm of the square of the norm of the Ricci curvature, and (one quarter of) the integral of the square of the norm of the Riemann curvature tensor, all these differences are constants independent of the Kähler metric choice (within a fixed Kähler class). So there is really only one minimization question.

This fact is not obvious, but it is not quite as surprising as might at first appear. Recall that if ω_1 and ω_2 are Kähler forms in the same cohomology class , then there is a global function such that the difference of the two corresponding metrics is the Levi form of the function. (This is just the so-called Hodge Lemma: If a real (1,1) form is d-exact, then it is $\partial\bar{\partial}f$ for some function f.) So all three of the apparently different minimization problems involve in fact one and the same function. Thus it is perhaps not so very surprising that the minimizations, which one thinks of as picking out a canonical one of the functions, should not depend on which minimization one chooses. This is of course not the proof! It just makes the observation plausible.

In the cases (negative and zero) where the Calabi Conjecture is always true (Aubin –Yau , Yau) , the minimization (for all three minimization problems) is given by the Einstein-Kähler metric with the given polarization (Kähler form).

Thus, the important and by now familiar Einstein Kähler metrics are obtained by minimizing the square of the curvature, analogous to the previous discussion of Riemann surfaces. But one can extend this to a more general bundle situation. This is the essential idea of Yang Mills theory.

Namely, in the most general possible terms, suppose one has a vector bundle over a compact Riemannian manifold which has some real (non-torsion) characteristic class that is not zero. Then it is not possible for the bundle to have a metric that is arbitrarily uniformly close to 0, since there is a formula involving the curvature for the characteristic class in question. Like the Euler characteristic in our initial discussion, or the first Chern class of the canonical bundle in the higher dimensional Kähler case, this characteristic class is topological—it does not depend on the metric chosen. (Of course, in the Riemann surface case, the first Chern class of the canonical bundle is the Euler characteristic, up to sign convention, so all dimensions are essentially the same situation.) So there is some definite restriction on how small the curvature can get. Clearly, under these circumstances, it is natural to look for minimization of the total curvature in some sense. The right sense usually turns out to be minimization of the integral of the pointwise squared norm of the curvature tensor in some form. (The square is motivated by our earlier example: it makes no real sense to minimize the integral of the curvature itself since that is, in our first example on the two-sphere, exactly the thing that is topologically fixed!)

To make this a bit more explicit, suppose that $E \rightarrow M$ is a vector bundle over a Riemannian manifold with a structure group G. (Typically, G will be an orthogonal or unitary group arising from a metric structure on the fibres of E). One can consider connections on E, that is operators that take local sections of E into E-valued one-forms on M, so that given a local section s and a tangent vector to M at a point p in M(around which the section is defined) , one gets an element of the fibre of E at p. One thinks of this element of the fibre as the derivative of the section along the tangent vector at p. This is required to satisfy a Leibnitz relation: in terms of one-forms $\nabla(fs) = f\nabla s + df \otimes s$, or ,if the vector is denoted by X,

$$\nabla_X fs = f\nabla_X s + (Xf)s .$$

Also, it is required that the connection ∇ is compatible with the group G in the following sense: Consider the ∇ parallel translation of elements of the fibre over a point p in M along a closed curve at a point p in M . This gives a linear map from the fibre at p to itself. This is required to be an element of the action by G on the fibre. This is required for all points p and closed curves at p.

Attached to the situation of a G-connection on a bundle is a curvature idea exactly analogous to the usual Riemannian geometry idea of the curvature tensor as the commutator of covariant derivatives, namely

R(X,Y) is an operator on local sections of V with values in local sections of V which is a 2-form (antisymmetric, bilinear in X and Y) on X and Y, defined by

$$R^{\nabla}(X,Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]}s \ .$$

This operator can be considered as a two-form with values at point in M lying in the Lie algebra of the group G acting on the fibre at that point. For example, in the Riemannian metric case, this would correspond to the antisymmetry of the curvature in its last two indices. In other words, the curvature in this case is an antisymmetric linear functional on the tangent space of M with values in the endomorphisms of the fibre of the vector bundle. i.e., if E is the vector bundle then the curvature is a section of $\operatorname{End} E \otimes \wedge^2 T^*M$. And the endomorphism values belong to the Lie algebra of G acting on the fibres of E.

To see how the Lie algebra arises, recall first the technique for proving directly that the Lie bracket is the commutator of flows: Let X and Y be (local) vector fields with flows f_t and g_t (so that, e.g. , $f_t(x)$, x in M fixed, t varies , is an integral curve of X). Then the difference between x and the composition $(g_{-t} \circ f_{-t} \circ g_t \circ f_t)(x)$ is order t^2 as t goes to 0 ; here the difference is computed as coordinate vector subtraction in any coordinate system around x. This difference turns out to the coordinate form of t^2 [X,Y] evaluated at x, up to higher order terms. This fact is established by a direct computation with Taylor expansions of the integral curves. In the present situation, something similar can be done. Namely, since R is a tensor in X and Y one can assume that the flows of X and Y commute. Consider parallel translation of an s in the fibre at x around the closed curve consisting of X-flow by t, followed by Y flow by t, then X flow by –t and Y –flow by –t. Again, a direct calculation with Taylor series that this has lowest order term quadratic in t as t goes to 0. And it coefficient is R(X,Y) s , in the sense that the translation of s around this curve is s plus $t^2R(X,Y)s$ + higher order terms. So parallel translation acts here on the fibre by identity plus the curvature item shown plus higher order. Now parallel translation around a closed curve gives an action on the fibre that is an element of G. It follows that the map that sends s to R(X,Y)s is an endomorphism of the fibre that must belong to the Lie algebra of G considered as mapped into the endomorphisms on the fibre at x via the action of G on that fibre.

This rather abstract description is a generalization of a familiar fact in surface theory, that the parallel translation of vectors around a smoothly closed curve that bounds a (small) disc in a surface differs from 2π-rotation (identity mapping) by the integral of the Gauss curvature over the disc. This is a special case of the Gauss formula, since the integral of the geodesic curvature is the rotation angle of the curve's tangent vector relative to parallel translation.

Overall, the curvature as Lie algebra valued for the vector bundle situation is a generalization of the familiar idea in Riemannian geometry that the curvature gives the infinitesimal generators of the (restricted) holonomy group as a sub-Lie-algebra of GL(dimM, R) (or in the Riemannian case of SO(dim M, R)).

Now suppose that M is a Riemannian manifold and E is given a (fibre) metric that is G-compatible. Then the differential forms with values in E and other natural bundles inherit metrics. In particular, there is a norm on tensors of the sort that the curvature tensor is. One defines the Yang Mills functional or "action" (terminology from physics) as $\int_M \left\| R^{\nabla} \right\|^2$ (or sometimes ½ this for physics reasons).

A Yang-Mills connection is by definition a connection that is a stationary (critical) point for this functional in the space of G-connections as described(this space makes sense: it is exactly that the first variation vanishes for all variations of the connection, as in classical calculus of variations). The Euler-Lagrange equation for this is computed to be that $\delta^{\nabla} R^{\nabla} = 0$ where δ^{∇} is the adjoint of the natural notion of covariant derivative operating on V valued forms. (This is defined exactly as in the coordinate –invariant formula for d, namely $d^{\nabla}(X_0,..X_R) = \sum(-1)^2 \nabla_{X_i} \alpha(X_0,..X_{k)}$ {with X_i omitted} + $\sum_{i<j}(-1)^{i+j} \alpha([X_i,X_J],X_0,..X_k)$ { with X_i and X_j }

The Bianchi identity is that $d^{\nabla} R^{\nabla} = 0$ already.
So the connection ∇ being Yang Mills is equivalent to R^{∇} being harmonic in the sense that R^{∇} belongs to both the kernel of d^{∇} d and its adjoint δ^{∇}. (Here one considers ∇ as a map into two –forms with values in the Lie algebra of G, as above, so that the adjoint goes from two forms to one forms with vaues in the Lie algebra.)

Thus Yang Mills theory is formally analogous to harmonic theory. (Recall that harmonic form theory is itself the solution to a minimization problem: the harmonic representative in a deRham cohomology class is exactly the unique form in that class that has minimal L^2 norm.)

In the case of the tangent bundle of a manifold of dimension four, this assumes special aspects. Since two forms are mapped to themselves under the Hodge star operator, one can consider whether * R^{∇} is a multiple of R^{∇} (which multiple would have to be +- 1 since * preserves length). This gives a decomposition of any R^{∇} into a "self-dual" and "anti-self dual" part, namely the eigenspace decomposition for the action of *. One has a lower bound from characteristic class theory in terms of the first Pontryagin class of V for Yang Mills functional and one sees that this lower bound is achieved by a connection if and only if it is self-dual or anti-self-dual (which one depending on the sign of the Pontyagin class).
In either case, self dual or anti-self-dual, R^{∇} is automatically harmonic (as n Hodge theory of the ordinary sort). This is all an analogue of the Hodge theory of the middle dimension forms on an even dimensional (oriented) manifold, dimension say 2k: * of an harmonic k-form is again an harmonic k-form so that the space of harmonic k-forms decomposes into spaces of forms on which * acts as identity or as minus the identity.

Hermitian Yang Mills theory is another specific realization of this very general idea of minimizing in classes which have a topologically determined lower bound on some functional , usually the integral of a squared norm . Namely, let us start with a holomorphic vector bundle over a compact Kähler manifold (it will turn out eventually that the Kähler condition is not really needed. But we want to start slowly). We can put an Hermitian metric on the vector bundle, more or less arbitrarily. Call this h and let g be the Kähler metric on the manifold.
Now it is a standard items that there is then a unique connection on the vector bundle with the properties:

1. the connection is compatible with the metric, i.e., parallel sections (along a curve) have constant length

and

2. the connection is holomorphic, that is, the connection forms relative to a holomorphic local frame in the vector bundle are type (1,0). This is the same as saying that the (0,1) part of Ds, for some local section s, is ∂s .

This is an easy thing to check:
Just take d $h(s_1, s_2)$, which must be $h(D s , s) + h(s_1 , Ds_2)$, and remember the conjugation in the second slot of h, then match up types of forms to get that the connection forms are the matrix of forms $h^{-1}\partial h$.

(see http://www.math.ucla.edu/~greene/Complex%20Differential%20Geometry.pdf page 47-48)

From here on , we denote the endomorphism –valued curvature tensor that arises as already described (from a given connection) by F.

Now, since we have a Kähler metric g, we can define a trace on F by contracting with the metric tensor, namely,
$\mathrm{tr}F = \sum g^{j\bar{k}} F^{\beta}_{\alpha j\bar{k}}$,
where Roman indices are for the manifold, Greek indices for the fibres of the bundle.

By definition, the holomorphic vector bundle is Hermitian Yang Mills over the Kähler manifold if there is an Hermitian metric h for which $\mathrm{tr}_g F=\mu \mathrm{Id}$ for some constant μ .

Note that this is an extension of the ideas of extremal Kähler metric (according to the Calabi ideas discussed above), the trace condition being in that case the Kähler-Einstein condition on the Ricci tensor(that it is a constant multiple of the metric). In turn, this implies that the Ricci curvature form (curvature in this Hermitian Yang-Mills sense) is harmonic since a constant multiple of the Kähler form is necessarily harmonic.

This harmonicity turns out to be equivalent to the metric being a critical point of a functional measuring in effect the integral squared norm of the curvature and thus it fits into the general picture already discussed. (In the Kähler Einstein situation of Yau, the Einstein Kähler metric critical point is, as noted, in fact an absolute minimum in the class of connection arising from Kähler metrics with the specified Kähler class.)

In case the bundle E is the holomorphic tangent bundle, then one can check easily that the Kahler metric is Hermitian Yang Mills if and only if it satisfies the Einstein-Kähler condition that the Ricci tensor (interpreted as a (1,1) form) is a constant multiple of the Kähler form, or equivalently the ordinary Ricci tensor is a constant multiple of the metric tensor g.

Note that for a Hermitian Yang Mills situation in general (E not necessarily the tangent bundle) , the constant μ can be computed in terms of the first Chern form of E and the Kähler form of the manifold metric g. Namely the first Chern form

$c_1(E) = \left(\frac{i}{2\pi}\right)\sum h^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}j\bar{k}} dz^j \wedge d\bar{z}^k$. From this, one obtains that

$$c_1(E) \wedge *\omega = \mu \sum h^{\alpha\bar{\beta}} h_{\alpha\bar{\beta}} \left(\frac{\omega^n}{n!}\right) = \mu \operatorname{rank}(E)\left(\frac{\omega^n}{n!}\right).$$

So $\mu = \left(\frac{1}{\operatorname{vol}(M)}\right)\frac{\deg(E)}{\operatorname{rank}(E)}$ where $\deg(E) = \int c_1(E) \wedge *\omega$.

At this point, one might want to refer back to the first paragraphs here to see how the general ideas persist: in general terms , the metric h is a stationary point for total squared curvature with the constant curvature being a topological item for the vector bundle, Kahler manifold combination.

The basic result of Uhlenbeck –Yau is this:
A stable vector bundle over a compact Kähler manifold admits a unique Hermitian Yang Mills connection.

Here stability is in the Mumford-Takemoto- Gieseker sense: the vector bundle E is stable if for each coherent reflexive subsheaf script F of the sheaf of germs of holomorphic sections of E,

degree (F)/rank (F) < degree(E)/rank E. Here degree is defined as $\int c_1(E) \wedge *\omega$.

Kobayashi had shown earlier that the existence of an Hermitian Yang Mills metric implies the stability of the bundle in this sense. The Uhlenbeck-Yau result (and its extension to the non- Kähler case by Li and Yau) is a converse to this.

Robert E. Greene