Summary of Profesor Yau's Lecture #7
Tuesday May 2,2007
Notes and supplementary remarks (in [ ]s) by Robert E. Greene

[Preliminary remarks on Levi forms and so on: Let M be a complex manifold with Hermtian metric $g_{i\bar{j}}$ in $(z_1,\ldots\ldots z_n)$ holomorphic coordinates. If f is a C^2 function on M (or on an open subset of M) then the Levi form of M is the Hermitian bilinear form

$$\sum_{i,j}(\partial^2 f/\partial z_i\partial\bar{z}_j)dz_i\otimes d\overline{z_j}$$

This has attached to it a trace item that is in effect a kind of Laplacian(for a Kahler metric this would actually be the Laplacian on functions), namely a function we shall call L(f)

$$\sum_{i,j}g^{i\bar{j}}(\partial^2 f/\partial z_i\partial\bar{z}_j)$$

This item is identifiable with a differential form item , namely (up to a constant factor) it is $(\partial\bar{\partial}f\wedge\omega^{n-1})/\omega^n$

In particular if f is a globally defined function then the integral $\int L(f)dvol=0$ by our usual method of moving ∂ and $\bar{\partial}$ over, provided that $\partial\bar{\partial}\omega^{n-1}$ =0 (ie that the metric is "balanced").]

We are interested in the version of the Bochner method that corresponds to the Levi Laplacian just described in the case of sections of holomorphic vector bundles with Hermitian metrics. Namely suppose that $V\to M$ is a holomorphic vector bundle with Hermtian metric h (which we shall eventually assume is Hermtian Yang Mills but this is not needed yet). At a given point p in M, we can find in a neighborhood a local holomorphic frame in V, say , e_1, ….. e_r , r= rank V, with the additional properties that The e's are orthonormal relative to h at p and that $d\,h_{\alpha\bar{\beta}}$ =0 at p for all index pairs α,β in 1,…r. Let s be a local holomorphic section of V in a neighborhood of p. We want to compute the Levi Laplacian of $\|s\|^2$ where the norm squared is computed relative to h. In particular, if $s=\sum s^\alpha e_\alpha$, then $\|s\|^2=\sum h_{\alpha\bar{\beta}}s^\alpha\overline{s^\beta}$. Using the usual formulas for the Hermtian connection of type (1,0) and the hypotheses about the frame , we get

$$L(\|s\|^2)=\sum g^{i\bar{j}}F^{\gamma}_{\alpha i\bar{j}}h_{\gamma\bar{\beta}}s^\alpha\overline{s^\beta}+\sum g^{i\bar{j}}h_{\alpha\bar{\beta}}(\partial s^\alpha/\partial z_i)\overline{(\partial s^\beta/\partial z_j)}$$

Each of these terms is the coordinate representation of a coordinate-free item. The first is the h inner product of the image of s under TrF with s itself. The second is the square of the norm of ∇s . In any case, the second term is nonnegative so if one has a condition ensuring that the first terms is everywhere nonnegative , the sum is nonnegative.

Such a condition arises in the Hermitian Yang Mills situation.
Namely, suppose that $\sum_{i,j} F^{\beta}_{\alpha i \bar{j}} g^{i\bar{j}}$ is c Id, where Id denotes the identity endomorphism on the fibre at the point and c is a nonnegative constant (independent of the point). Then the first term of the Levi Laplacian just given is nonnegative at each point while, as noted, the second terms is always nonnegative(independently of curvature assumptions). In this case, the fact that $\int L(f)dvol = 0$ as already discussed implies that both terms vanish at each point.
In particular, this shows that if the curvature is Hermitian Yang Mills with Yang Mills curvature (the trace of F as given) nonnegative then any nontrivial holomorphic section must be "parallel", that is a covariant constant. And if there is a nontrivial holomorphic section then the nonnegative constant c must in fact be 0.

[This of course is a generalization of a familiar idea about line bundles, saying that a negative line bundle has no nontrivial holomorphic sections and if a nonpositive bundle has a nontrivial holomorphic section, then the bundle is flat and trivial and the nontrivial section is a constant length and globally parallel relative to the flat connection.]

As usual in the Bochner technique, one can also think of the whole matter in terms of the Maximum Principle for the second order elliptic operator , the Levi Laplacian.

Now this idea has a wider utility than might at first appear. To begin with, the existence of a metric on the bundle that is Hermitian Yang Mills (in our usual sense that TrF is a constant times the identity)as required for the argument just given is guaranteed by stability (relative to a "polarization", that is, a choice of Kahler form ω with $\partial\bar{\partial}\omega^{n-1} = 0$).
And the sign of the constant c(or its being 0) is computed topologically, namely, by (1/rank(V)) $\int c_1(V) \wedge \omega^{n-1}$. For example, if this integral is 0 (and V is ω-stable) , then every holomorphic section of V is parallel.

Now from a given holomorphic bundle, one can construct in a natural way other holomorphic vector bundles. Think of the original bundle as built from $U_\lambda \, x C^r$, with the U's being a trivializing cover for M relative to V(ie V is holomorphically trivial over each U) by gluing together the copies of complex Euclidean r-space by a holomorphically varying element of Gl(n,C) assigned to each point of the intersection of two of the U's. This map into Gl(n,C) gives a map into the complex general linear group acting on tensor powers of C and C^* (the dual of C) ,that is $(\otimes^p C^r) \otimes (\otimes^q (C^r)^*)$. This tensor product can be decomposed into irreducible subspaces for the induced action from Gl(n,C). (These are obtained from symmetry properties using Young diagrams and so on). Now each of these irreducible subspaces gives rise to a holomorphic bundle. (Note:All of these except for $\wedge^n C$ and its dual have trivial determinant bundle.)

Now if the original V was stable and hence admits an Hermtian Yang Mills metric and connection, then so do each of these new tensor-derived bundles. This is easy to see by computing the connections induced on the induced bundles from the connection on V. Since the existence of an Hermitian Yang Mills structure on a bundle implies its stability(result of Kobayashi) this means that all these derived tensor-type bundles are also stable. This is far from obvious using only the definition of stability in terms of degree and rank of sub-sheaves that we introduced earlier!
(ie deg F/rank F < deg E /rank E for every reflexive sub-sheaf of E is the condition for stability of E). The Hermitian Yang Mills characterization of stability gives a neat proof here of the stability of the tensor-constructed bundles.

Now many of these bundles will satisfy the condition that the associated constant c in our previous notation is nonnegative. From this one deduces that many of these bundles have all holomorphic sections parallel (or if the associated c is positive , that no section is holomorphc except the zero section).[This is of course familiar in Riemann surface theory. E.g., if the canonical bundle is negative then so is the bundle of quadratic differentials etc. and hence the only holomorphic quadratic differentials are zero, explaining why the Riemann sphere is deformation rigid, while the quadratic differentials on a torus have a parallel nonzero holomorphic section and every holomorphic section is a constant multiple of that one, explaining why the moduli space is complex dimension 1. etc.]

The existence of such a parallel holomorphic section of one of the tensor –derived bundles actually shows that the holonomy (identity component) of the original bundle reduces to a smaller group (than U(n)). The simplest example is that if there is a fixed element(a parallel element that is, so that it is fixed under holonomy) for $\wedge^n C^n$, n = dimension of M, then the holonomy of the tangent bundle of M must reduce not just to U(n) (on account of the metric) but to SU(n).

This kind of argument seems to depend on exactly which one of the tensor-derived bundles on considers. While they are all stable as noted, the sign of the constant c associated (or its zeroness) would seem to require examination in each case. But in fact as long as one is not looking at $\wedge^n(C)$ or its dual, then in fact the constant c is always 0 so that the argument of Bochner-type holds and every holomorphic section is parallel. So if one assumes that there is one of the tensor –derived bundles that is different from $\wedge^n$ or its dual but that has a holomorphic section, then the holonomy group of the original bundle is reduced.

With V= the holomorphic tangent bundle,assumed stable, and if the connection is torsion free(which is the same as the metric being Kahler: the torsion of the Hermitian type (1,0) connection is 0 if and only if the Hermitian metric is Kahler), then the holonomy groups that can occur are classified. If the group is reduced by a holomorphic tensor other than $\wedge^n$ or its dual, then the manifold is a (locally) symmetric space of higher rank than 1. If the group is SU(n) , then the manifold is Calabi-Yau. Note that this is if and only if: if the holonomy is reduced, there must be an invariant tensor and this will a parallel holomorphic section of one of the tensor –derived bundles.

In other words: With the tangent bundle assumed stable, we have that the manifold is a locally symmetric space of higher rank if and only if there is a tensor-derived bundle (other than $\wedge^n$ or its dual) with a nontrivial holomorphic section.

This makes possible a surprising application to what otherwise appears to be apurely algebraic question. Namely, look at a Shimura variety(Hermtian locally symmetric space of rank .1) embedded in complex projective space via automorphic forms and defined then by the vanishing of a set of polynomials with coefficients in an algebraic number field. The Galois group of the number field operates on these polynomials and gives a new manifolds as the vanishing set of the new set of polynomials. Such a new manifold is called a Galois conjugate of the original Shimura variety. This conjugate can be quite different from the original variety. For example, their fundamental groups can be different. D. Kahzdan proved (difficult theorem) that a Galois conjugate of this sort is again a Shimura variety. Note that this follows from the characterization we have obtained! since everything in the characterization of Shimura varieties(Hermitian locally symmetric of rank >1) is natural under the Galois action!!!

We have been working primarily with complex manifolds but of course it is natural and important to try to do similar things with general, real manifolds which may not have any complex or even almost complex structure.

The most familiar and to date the most successful application of curvature-minimization ideas has been to 4-manifolds, in the form of the work of Donaldson and subsequently others using Yang Mills theory and the self-dual and anti-self dual curvature ideas. The basic idea was outlined earlier [in “After the Interlude”], in terms of minimization of curvature norm squared integrals for complex vector bundles. [Such vector bundles arise in a natural way. For example, in the case of a spin structure, one finds natural bundles with structure group SU(2) , associated to the homomorphism of SO(4) into SO(3) x SO(3) via the SO(4) action on the bundles of self-dual and anti–self-dual 2-forms, and the local isomorphism of SO(3) to SU(2), SU(2) being the (Lie group) double cover of SO(3).] The crucial idea here is the use of the moduli space. Information is obtained by what amount to singular perturbation techniques of bundles defined at one point, that is , gluing in a points anti-self dual connections on bundles over S^4 and then perturbing these to be anti-self-dual over the whole space. This process produces a wealth of topological information. However, the process depends upon in effect knowing the bundle quite well. To extend it to more general, less specified bundles, one needs to have some general idea that would correspond to stability in the complex case.

A particular utility of the Donaldson approach arises in the complex case, that is the case where the (real) 4-manifold is itself complex. Because self-duality is not defined in terms of the manifold’s complex structure, but depends only on its differential-topological structure, the invariants constructed are independent of the complex structure and are differential-topological invariants. But they can then be used to give information about the complex structure. Similar considerations apply to the Seiberg-Witten invariants(which may be equivalent to the Donaldson invariants in some way: this is unknown at present). This has led to some remarkable results on the diffeomorphism possibilities of complex surfaces.

For example, start with a complex surface M with Kodaira dimension >0. (These include general type and some other algebraic surfaces). This can be shown not to be diffeomorphic to a rational surface. (A rational surface is by definition the result of blowing up a finite number of points of CP^2 or CP^1 x CP^1). We shall discuss this striking result later via Seiberg Witten invariants. It is natural to ask whether this can be extended to homotopy equivalence. This cannot be proved from Donaldson or Seiberg Witten invariants since these are diffeomorphism invariants only.

One would like to extend these ideas to situations more general than Kahler manifolds. Kahler is advantageous because it gives self –dual objects so some difficulties arise in extending.

Returning to the homotopy type situation, note that if an algebraic surface M is the homotopy type of CP^2 then it is diffeomorphic to CP^2. But this is not proved by Seiberg Witten nor Donaldson invariants, which would need diffeomorphism to begin with.
In general , one can ask how much of the diffeomorphism results aer really homotopy type results.

That there is some subtlety here follows from considering the situation of Kodaira dimension 0(K3 surfaces, Enneper surface, abelian varieties, and quotients). In this instance, Kodaira showed that homotopy type was not enough in a sense. Specifically, he began with a K3 surface which was an elliptic fibre space over CP^1 with some singular fibres(as typically happens, most cases being like this) and showed how to cut out a tube around a singular fibre ,apply the so-called "log transform" and glue back in to form a new algebraic surface. This surface has the homotopy type of the original K3 but has $c_1 \neq 0$. (We omit the details of the log transform, but it is important to know that such things are possible.)

There are very interesting questions remaining in this direction of homotopy type. For example, it remains unknown whether a surface of general type that is of the homotopy type of CP^1 x CP^1 is diffeomorphic to CP^1 x CP^1 . (By Friedman, homotopy type here is the same as homeomorphism.)

For these questions, one needs a more general method than those presently available.

Later on we shall discuss in some detail the extension of various ideas of complex manifolds to the symplectic case. (As usual, a symplectic structure is by definition a closed 2-form ω on a manifold of real dimension 2n with $\omega^n = 0$; one says that ω is

“non-degenerate”). We shall be interested in the ideas initiated by Gromov of pseudo-holomorphic curves and related questions.
The set-up here is that one thinks of almost complex structures J on M that have a certain compatibility with the symplectic form, namely that ω(X, JX) >0 for all nonzero X in the tangent space at a point, for all points. Such an almost complex structure is sometimes called “tame” with respect to the symplectic structure. It turns out that such almost complex structures always exist and form a contractible set. (They are not unique since the condition is an open one in the space of endomorphisms of the tangent bundle so that a small perturbation preserves tameness). Fixing one such almost complex structure on a symplectic manifold, one can try to generalize Kahler geometry to this “almost Kahler” situation.

In particular, one can talk immediately about many familiar concepts. For example, a map of a compact Riemann surface C into M, say $f:C \to M$, is said to be pseudoholomorphic if the diffential of f (as a real mapping) commutes with J:
$df \circ J_C = J_M \circ df$, where J_M is the fixed tame almost complex structure on M.

Gromov has shown :
Suppose M is diffeomorphic to CP^n and that there is a “rational pseudo-holomorphic curve”(of degree 1, i.e., the curve generates the 2-cohomolgy of CP^n) Namely ,suppose that there is a map of CP^1 into M that is pseudo –holomorphic in the sense indicated with the cohomology generation property indicated.
Then M is actually symplectically diffeomorphic to CP^n (with its standard symplectic structure arising from the Kahler form of its standard Kahler metric). In other words there would be a diffeomorphism of CP^n to itself such that the pullback of the standard symplectic form would be the given symplectic form.

One says for short in this case, that there is (would be) only one symplectic structure on CP^n.
(This is a famous question, the uniqueness of symplectic structure on CP^n , n>1. The n=1 case is automatic since Uniformization of Riemann Surfaces applies)

But the construction of a suitable pseudo-holomorphic curve is apparently very difficult. Taubes proved the first major result: Such a pseudo-holomorphic curve exists when n=2 .

This line of thought is similar to the Siu-Yau proof of the Frankel Conjecture, where the construction of a rational curve was also crucial.

It remains unknown whether a compact symplectic manifold of real dimension 4 that is of the homotopy type of CP^2C is diffeomorphic and hence symplectically diffeomorphic to CP^2 . (As pointed out earlier, this is true for algebraic surfaces, but it is unknown in the symplectic case.)