Building Geometric Structures
Summary of Profesor Yau's Lecture 2
Thursday, April 5,2007
Notes and supplementary remarks (in [ ]s) by Robert E. Greene

Last time:
General idea: build geometric structures using methods of (usually nonlinear) partial differential equations. Prototypical/motivating case: try to build a complex structure (local coordinate systems with holomorphic transition functions) from an almost complex structure. In (real) dimension 2, there is no obstruction to this, no "integrability condition". The reason is that the general condition , $\bar{\partial} \circ \bar{\partial} = 0$,is automatically satisfied in this case because of dimension conditions. In general, $\bar{\partial} \circ \bar{\partial}(\Omega)$ for any Ω will have q degree in terms of (p,q) types at least 2 and hence will necessarily be 0 in the real dimension 2 case. Thus the general Newlander –Nirenberg condition is always satisfied in the real dimension 2 case. Note: $\bar{\partial} \circ \bar{\partial} = 0$ condition is equivalent , by tensor calculation , in any dimension to the more usual condition of vanishing of the Nijenhuis tensor N, defined by

$$N(X,Y)=[X,Y] + J([JX,Y]) + J([X,JY]) - [JX,JY] .$$

[The real dimension 2 case can be done without Newlander-Nirenberg by using isothermal parameters of Korn-Lichtenstein et. al. –see notes for lecture 1].

The Newlander-Nirenberg Theorem is fundamental because it does the smooth , C^∞ case. The result was known before in the real analytic , C^ω category. This comes from Cartan-Kähler theory for exterior differential systems, which gives a general result in the real analytic category case. E.g, in the real analytic situation, if the tangent bundle admits a G-connection, torsion-free, then there are charts for a G structure[in the sense that the transition functions have tangent-space maps, that is differentials, that belong to G at each point so that the reduction of the structure group of the tangent bundle to G is in fact realized in coordinate charts. This corresponds, in the reduction of Gl(2n , R) to G= Gl(n, C) case, to finding a set of charts with holomorphic transition functions, because a (smooth) function on an open set in R^{2n} to R^{2n} is holomorphic if and only if its real differential is complex linear, namely, its differential commutes with the almost complex structure tensor J.]

The deformation parametrization of the complex structures on a compact Riemann surface Σ_g of genus g is by quadratic differentials, that is holomorphic sections of the square of the canonical bundle, i.e. $H^0(\Sigma_g, K^2)$, K= the canonical bundle= (in this dimension) the bundle of (1,0) forms. [See notes for lecture 1]. For complex projective structures, which will be discussed in more detail later, the corresponding item is $H^0(\Sigma_g, K^3)$.

Returning now to the situation of complex manifolds and almost complex structures, we look at this question:

Suppose $V \mapsto M$ is a complex vector bundle with complex fibre dimension k. So the transition functions have values in Gl(k, C) which are a priori not necessarily holomorphic functions on the the overlaps of trivializing open sets in M.

Question: Is it possible to choose trivialization (relative to some open cover) that do in fact have the transition functions holomorphic?

In other words, can every topological(or smooth) complex vector bundle be realized as a holomorphic vector bundle?

In the general setting, there is an integrability condition again when trying to apply the Newlander Nirenberg Theorem here. Namely, one would like to apply the Newlander – Nirenberg Theorem to an almost complex structure on the total space that induced the complex structure already given on each fibre(complex vector space). Finding an integrable such almost complex structure is equivalent to the existence of a connection on the complex vector bundle which has curvature of type (1,1). That is, the (0,2) and (2,0) parts of the curvature form need to be 0.

Again, this turns out to be automatic in the case that M is a Riemann surface[For line bundles, this is much more elementary: Line bundles L are determined topologically by their first Chern class c_1 (L), which is in the integer 2-cohomology of the Riemann surface. This cohomology is isomorphic to Z. Moreover, there is a holomorphic line bundle L with $c_1(L)$ corresponding to a generator of Z. It suffices to take the line bundle associated to a divisor with one point , coefficient=± 1. Then the tensor powers (positive, negative or zero) of this L realize all topological possibilities, since their first Chern classes account for all of the second integer-coefficient cohomology.]

So in the Riemann surface case, every complex line bundle has a holomorphic representative.

This viewpoint is useful for analyzing the following situation: Suppose M is a Kähler manifold and we are looking for maps of $\Sigma \mapsto M$ that are holomorphic, Σ a Riemann surface. That is , we are looking for holomorphic curves in M.

In this situation, the holomorphic curve has the special property that it is area-minimizing (in the induced metric) within its homology class. This is called the “Wirtinger inequality”. It is easy to prove but important.

The proof consists of noting that if ω is the Kähler form of M, ω is closed and $\|\omega\| = 1$.

Thus the norm of w pulled back to any real-two-dimensional submanifold Σ' (not necessarily complex!) is also less than or equal to 1. If Σ' is in the homology class of Σ, then since ω is closed $\int_{\Sigma} \omega = \int_{\Sigma'} \omega$.

But then $\|\omega\| \leq 1$ on Σ' gives $\int_{\Sigma'} \omega \leq area(\Sigma')$ so the proof is complete.

(This gives rise to the idea introduced by R. Harvey and B. Lawson of a "calibrated geometry": Suppose Ω is a form that is closed and has norm$\leq$ 1.
Then if F:N$\mapsto$ M has $\|F^{*}\Omega\| = 1$, then N is volume minimzing in its homology class by the same argument. The condition that $\|F^{*}\Omega\| = 1$ is automatic in the Kähler situation, with Ω = a power of the Kähler form and F a holomorphic embedding or immersion.)

A good guiding principle in general is to look at the converse of statements to see when it holds. So a natural thing to do here is to look at when the converse of the Wirtinger idea applies. [Note: One kind of converse is more or less immediate: If a 2- homology class has a holomorphic representative ,which is consequently absolutely minimzing of volume with the class, then any other absolute minimizer in the class is necessarily holomorphic or anti-holomorphic. This follows from just observing that if the chain of inequalities in the proof are all equalities then the pullback of the Kähler form must have norm 1 at every point. And this is possible only if the embedding is holomorphic or anti-holomorphic.] The interesting/challenging kind of converse is to try to find conditions under which the area minimizer in a class is necessarily holomorphic or anti-holomorphic.

In this situation, we start with Σ_g and with a smooth , but not necessarily holomorphic map(embedding or immersion) F:Σ $\mapsto$ M, into a Kähler manifold M. And suppose that this is area minimizing in its homology class. (Actually, we shall need only that it is a local stable minimum, as we shall discuss momentarily). Then the first variation must be 0, so that Σ as a submanifold of M is a minimal surface. Then we look at the second variation. Just as the second order part of a Taylor series must be non-negative definite at a local minimum of a function (since f(t)= f(0) + t f(0) + (t^2/2) f (0) + …) so the second variation must be nonnegative for any variation of the map F. Namely, using the usual I notation for the index form,

$$0 \leq I(s,s) = \int \|\nabla s\|^{2} + \text{curvature term}$$

where s is any vector field normal to F. The "index form" is in effect the second derivative of the area function.

For real co-dimension 1 [in the general minimal surface case; this cannot happen in the Riemann surface into Kähler manifold case], things are simplified because there is no normal connection, since there is (locally) a unique unit normal up to$\pm$. Thus one can simplify the ∇s term. Most arguments about minimal surfaces are like this and apply in this case. Write s =fN, where N is the unit normal. Then the first term is just $|df|^2$ and $I(s,s) = \int |df|^2 + f^2 \text{Ricci}(N,N)$.

If one takes f=1 identically then I becomes $\int \text{Ricci}(N,N)$ only.

For instance, one shows that if the Ricci curvature is positive, then no (real) hypersurface is stably minimizing, i.e., a compact manifold with positive Ricci curvature has no stable minimal hypersurface. This implies that the first Betti number is 0. (well known otherwise) [Other proofs: Bochner technique shows that a harmonic 1-form must vanish identically. Alternatively, Myers' Theorem –complete manifold with Ricci bounded below by a positive constant is compact--implies that the universal cover is compact so that the fundamental group is finite and hence again first Betti number is 0.]

Higher co-dimension: In general there will not be a parallel section of the normal bundle(existence of such would imply that the holonomy in the normal bundle reduces and in general this is not the case), i.e., there is no "covariant constant" normal section and one is stuck with the ∇s term in the index form.

Complex case: I(s,s) = $\int \left|\bar{\partial} s\right|^2 + \text{curvature term}$

Now look at the pullback of the tangent bundle of M to Σ, namely the complex(but probably not holomorphic) vector bundle . Use that any complex vector bundle over a Riemann surface can be made holomorphic. Then use the Riemann-Roch Theorem (under suitable conditions) to get a section s with $\bar{\partial} s = 0$.

This technique was first used by Siu-Yau (1980) to prove the Frankel Conjecture [that compact Kähler manifolds of positive bisectional curvature were biholomorphic to CP^n]. Namely, if bisectional curvature is positive, then a stable minimal surface is holomorphic or antiholomorphic. This gives a technique to get rational curves.(Here one uses that every complex vector bundle over the sphere is equivalent to a sum of line bundles to get normal sections). The role of the curvature hypothesis is that with s chosen as in the previous paragraph, the curvature term will be negative unless the surface is holomorphic on anti-holomorphic.

The result of Micallef-Moore on positive curvature operator and positive isotropic curvature is proven by using some of the same ideas:

The ideas here are an extension of the stable minimum idea. Here one gets not positivity of index but rather control of the dimension of the maximal negative subspace, i.e., the Morse index. Namely, look at the maps of S^2 into a Riemannian manifold M and consider the index form I as before. This is a quadratic form on variation vector fields. Look at the maximum dimension of a subspace on which the form is negative definite(=union of the negative eigenspaces). This (finite) dimension is the Morse index by definition. Note that the positive part is infinite dimensional. It is easy to increase area[e.g., by making things "bumpy", with large values of the derivative of the normal variation vector field] . "There is much more freedom to go up than down".

Micallef-Moore control Morse index under assumption of positive isotropic curvature of M. [This is a curvature condition which is implied by the more usual condition of positive curvature operator, that is , the curvature tensor R being positive as an operator from wedge 2 of the tangent space to itself]. Positive isotropic curvature is defined by first extending the inner product to the complexified tangent space (no conjugation to make an Hermitian metric , straight complex linear extension). Then look at "isotropic" complex vectors, i.e.,v= X+iY such that the complexified inner product of this with itself is 0, $\langle v,v\rangle = 0$. Then what is required for PIC(positive isotropic curvature) is that for any two such isotropic vectors v,w that span a complex 2-plane containing only isotropic vectors, $\langle R(v,w)\overline{w},\overline{v}\rangle$ is positive .

The general idea of M-M is that positivity of curvature should imply that minimal surfaces (or more generally minimal submanifolds) tend to have more directions in which the index is negative. [This is of course an idea that goes back much further, e.g., Synge's Theorem that a compact even dimensional orientable manifold of positive sectional curvature must be simply connected, which is proved by constructing a shrinking of a geodesic of minimal length in a given nontrivial free homotopy class of curves, a contradiction which shows that the manifold has trivial free homotopy and is hence simply connected.] A high dimensional sphere is an example: a standardly embedded 2-sphere has (at least) an n-2 dimensional space of deformation fields with negative second variation of area, namely, the space of parallel normal vectors.

To get the index estimate, one needs to find deformation vector fields with negative second variation. This is done by complexifying the tangent bundle of M and pulling it back to S^2 via the map into M. Using the result noted, that this bundle can be made holomorphic and also the result(Grothendieck) that a holomorphic vector bundle over S^2 splits as a sum of holomorphic lines bundles, one can use the Riemann-Roch Theorem to find holomorphic sections of the pullback bundle. These give rise to isotropic deformation vector fields for F which have negative second variation (on account of the PIC condition). This method is derived(as acknowledged by M&M) from the similar situation that had been treated already in Siu and Yau's proof of the Frankel Conjecture(see below).

Thus under the particular curvature assumption(PIC) , M-M show the Morse index of any 2-sphere minimally embedded in the manifold M^n , n> 3, has index at least n/2 -3/2.

Now Morse theory more or less works in this setting (more detail in a moment), and one expects that the Morse theory (assuming simple connectivity) of C^∞ (S^2, M) gives homotopy information that will give the homotopy of M shifted by two levels(just as the Morse theory of the loop space gives the homotopy of M shifted by one level). Carrying this through yields that the homotopy groups of M are zero up to and including dimension [n/2]. By Poincaré Duality, this gives that M is a (homotopy) sphere and hence homeomorphic to a sphere.
(This is about manifolds of dimension at least 4 and hence does not use the 3-dimensional Poincaré Conjecture).

[This result is closely related to the historic Berger-Klingenberg Quarter Pinched Sphere Theorem as well as to results of the Bochner technique. The B-K Theorem is that a compact simply connected manifold with sectional curvature strictly between ¼ and 1 is homeomorphic to a sphere. Now it is (pointwise) algebraic that this "quarter pinching" implies PIC, but there are PIC curvature tensors that are not quarter pinched (even up to scaling). So the BK Theorem in dimension 4 or greater is implied by the (later) Micallef-Moore result. (Strictly speaking, this would be true only up to homotopy, but by now the Poincaré Conjecture is known in all dimensions so this distinction has become moot). Actually, since the quarter pinching implies PIC is point-wise, a stronger result is obtained: one needs only sectional curvature between A/4 and A where A may depend on the point. Recently, Brendel and Schoen have shown that such pointwise quarter pinching in fact implies that the manifold is diffeomorphic to the standard sphere. This was previously unknown even if the A involved(previous notation) was independent of the point. Since ¼ is optimal (e.g., the standard metric on complex projective space), this result brings to a conclusion this whole line of investigation of pinched positive curvature implying resemblance to the standard sphere. This pointwise quarter –pinched diffeomorphism result improves results of Ruh(pointwise but dimension dependent) , Sugimoto-Shiohama(uniform but independent of dimension, larger constant than ¼), and many others, where various stronger pinching conditions were required. Historically, these questions were also approached from the viewpoint of the Bochner technique. This technique gives, in a result of Meyer extending the result of Bochner-Yano, which required ½ -pinching of eigenvalues, that positive curvature operator implies vanishing of all Betti numbers except the nth (and 0^{th}). But this ,while an interesting confirmation, is not sufficient to give the whole homeomorphism, homotopy, or diffeomorphism - type results. (This result was reproved by Poor, using ideas of Chern on the relationship between holonomy and the Laplacian.) In dimension 2, everything is automatic from classification of surfaces, and in dimension 3, the Poincaré Conjecture disposes of the situation. Historically, Hamilton proved for three dimensions that positive Ricci curvature and simply connectivity implied diffeomorphism to S^3, prior to the Poincaré Conjecture proof.]

Making Morse theory work:

Idea of Sachs-Uhlenbeck: Look not at the usual energy function of harmonic map theory but instead at the α-energy, $\alpha>1$, namely, by definition, $\int\left(\|df\|^2+1\right)^{\alpha}$

If $\alpha=1$, this is the usual energy situation. But $\alpha>1$ works better analytically. In the α greater than 1 case, one gets the classical condition for Morse theory, "Condition C" of Palais/Smale so that Morse theory works in the usual way. Then one takes a limit as α goes to 1.

This makes the Micallef-Moore program described above literally work, One gets homotopy groups to be zero up to [n/2] , which suffices to get a homotopy sphere (use Poincare duality). Specifically, M-M use the method of Sacks-Uhlenbeck to show that if the homotopy groups of M are 0 up to level k but the kth group is nonzero, k at least 2, then there is a nonconstant minimal (harmonic) two-sphere of index not greater than k-2. Combining this with the index estimate above(index at least n/2 -3/2) shows that k is at least [n/2] +1. That is , the homotopy groups are 0 up to and including [n/2]. And, as already noted above, this suffices via the Hurewicz Theorem combined with Poincare Duality to show that M is a homotopy sphere and hence is homeomorphic to a sphere.

Generalization to situations with weaker curvature conditions and also higher dimensions.

This generalization is not entirely possible but it is desirable to the extent it is possible.

Approach (for the complex case) via integrability of complex bundles as discussed briefly already.

Siu/Yau: Compact Kähler manifold with bisectional curvature >0 is necessarily biholomorphic to CP^n(Frankel Conjecture)

[Bisectional curvature: If X and Y are unit vectors that are perpendicular then B(X,Y)= -R(X,Y, X,Y) – R(X,JY, X,JY). This depends only on the span of X and Y. Positivity of this is intermediate between positivity of all sectional curvature—it is a sum of sectional curvatures—and holomorphic sectional curvature--- B(X, JX) = the sectional curvature of the J invariant 2-plane spanned by X and JX , since in this case the second term in the definition of B vanishes.
A compact Kähler manifold with positive holomorphic sectional curvature, and hence of one with positive bisectional curvature, is simply connected, by an argument almost exactly along the lines of the proof of Synge's Theorem: If c is a smoothly closed arc-length-parameter geodesic in the manifold, then J applied to the tangent vector c' of c is a parallel unit normal along c which is smooth, including at the (nominal) endpoints of c. Deformation of c along this normal field has negative second variation since the curvature term in the second variation formula is the negative of the holomorphic section curvature of the plane spanned by c' and Jc' , the term involving the covariant derivative

of the deformation vector field being zero in this case. This is a contradiction if c is the minimal curve in a nontrivial free homotopy class. Since positive bisectional curvature implies positive holomorphic sectional curvature, the manifolds involved in the Frankel Conjecture are automatically simply connected. Thus the fact that H_2 is nonzero makes π_2 also nonzero so that there are nontrivial homotopy classes of maps of S^2 over which one might hope to minimize. This would of course not necessarily be the case without the simple connectivity, e.g., a complex torus. Alternatively, one could detour around this variant of the Synge's Theorem argument in proving the Frankel Conjecture by passing to the simply connected cover, which is necessarily compact by Myers Theorem, since the Ricci curvature is positive if the bisectional curvature is. But if the universal cover is biholomorphic to CP^n then so was the original manifold since CP^n has no complex quotients. So it suffices to prove the result under the explicit assumption of simple connectivity. That CP^n has no complex quotients follows easily from e.g., the Lefschetz Fixed Point Theorem: With the Kaher metric normalized so that the cohomology class of the Kähler form is a generator of the second cohomology with integer coefficients, the pullback of the cohomology class of the Kähler form by a biholomorphism again has this property and hence is +1 or -1 times the class of the original Kähler form. Checking on X, JX shows the multiple is +1 from which it follows that the Lefschetz number of the map is (n+1). So the bihomorphic map has a fixed point and hence cannot be a nontrivial covering transformation.]

More general result than Siu/Yau on matter related to the Frankel Conjecture :

Mori: If the tangent bundle of a compact Kähler manifold is positive, then M is biholomorphic to CP^n.

Mori's work includes more information on construction of rational curves and gives rise to structural results, even with just nonnegativity conditions . [for more on "Mori theory", cf. e.g., Campana and Peternell, Recent Developments in the Classification of Compact Kähler Manifolds, *Several Complex Variables*, MSRI Pub. 37(1999) available on-line]

The crucial point in the Frankel Conjecture is to find a rational curve. [Then restriction to it of the tangent bundle ,which splits as a sum of line bundles, makes it possible to apply a characterization of CP^n by Kobayashi-Ochiai to obtain the final result: see Siu/Yau Inventiones Math. 59,1980 for details]. This rational curve is obtained by the method already discussed. (Siu/Yau argument came first and was motivation for M-M). It is possible to estimate degree of rational curve here, same as estimating area, $\int \omega \leq n$ (special argument for this case).

Philosophical difference between Siu/Yau and Mori:
Siu/Yau is Kähler geometry, Mori is really not, but rather in effect Finsler geometry. Namely, positivity of the tangent bundle is equivalent to the existence of a Finsler metric with positive curvature.

The idea of finding rational curves has a symplectic analogue, developed by Gromov. The idea is to create a map analogous to a rational curve, a "pseudoholomorphic curve". Gromov showed that if one such existed then one got a rigidity result, that a symplectic structure on a manifold homeomorphic to CP^n had a pseudoholomorphic curve, then that manifold with that structure was symplectomorphic (diffeomorphic by a symplectic map) to CP^n with the standard structure.

Whether this always happens(whether such a pseudoholomorphic curve always exists) remains an unsolved problem. Taubes proved an affirmative answer if the symplectic manifold is diffeomorpic to CP^2.

Return to almost complex structure questions:

Complex dimension 2: Suppose a real 4-manifold has an almost complex structure. This happens if and only if there are cohomology classes, a 2-class c_1 and a 4-class c_2 , with the necessary relationship to the second Stieffel-Whitney class and the first Pontryagin class of M[see notes for lecture 1]. Then we want to know if there is an associated complex structure.

Note first that there is the Riemann-Roch formula of Hirzebruch:

If $V \mapsto M$ is a holomorphic vector bundle and if $H^i(M, V)$ denotes the sheaf cohomology for the sheaf of germs of local holomorphic sections of V, then

$$\sum (-1)^i \dim H^i(M,V) = \int ch(V)Todd(M)$$

where the righthand side is a purely topological item involving characteristic classes (ch(V) is the Chern character, Todd(M) is the Todd class of M).

From the Atiyah-Singer Index Theorem , this is true even in the non-Kähler case.

Now one needs to realize something very simple but useful:

In complex dimension 2, there are only two terms on the left hand side with a +, namely $H^0(M,V)$ and $H^2(M,V)$ and the second one of these is the same dimension as $H^0(M,V^*\otimes K)$. So we get

$\dim H^0(M, V) + \dim H^0(M,V^*\otimes K) = \dim H^1(M,V) +$ a topological item

and the right hand side is thus greater than or equal to the topological item, namely

$\int ch(V)Todd(M)$. [This is the two-dimensional analogue of what is known in Riemann surface theory with V a line bundle as the Riemann inequality, in which case the topological term is just 1-g + degree (V), where degree(V) is the integrated Chern class of V].

In most instances, one is interested in V built from the holomorphic tangent bundle or its dual via exterior products. One then uses topological information to get holomorphic sections of either V or $V^* \otimes K$.

This gives obstructions to the existence of integrable almost complex structures.

Work of van de Ven: For a compact complex manifold of complex dimension 2,
$8c_2 \geq c_1^2$
This implies that there are many almost complex 4-manifolds with no complex structure.

Yau [Topology, 1976] gave examples of real four-dimensional manifolds with topologically trivial tangent bundle and hence with all Chern numbers 0 which admit no complex structure [event though they obviously have an almost complex structure!]. This is not possible if the manifold is simply connected since the Euler number being 0 in this case implies that the first Betti number must be nonzero [the first Betti number equals the third by Poincaré duality, so if the first Betti number were zero, there would be no negative terms in the Euler characteristic sum and the Euler characteristic would thus be at least 2]. [Both the van de Ven and Yau results depend on information from the classification of complex surfaces by Kodaira and are thus specific to the complex dimension 2 case].

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