Sample Problems for Midterm I (Monday, April 27, 2009)

1. Suppose (X, d) is a complete metric space and $\overline{B}(p_i, r_i)$ $\left(\overset{\text{def.}}{=} \{x : d(x, p_i) \le r_i\}\right)$ is a sequence of closed balls with $\lim r_i = 0$ and $\overline{B}(p_{i+1}, r_{i+1}) \subseteq \overline{B}(p_i, r_i)$ for each $i = 1, 2, 3 \ldots$ Prove that $\exists\, p_0 \in X$ with $p_0 \in \bigcap_{i=1}^{+\infty} \overline{B}(p_i, r_i)$.

2. Prove that a compact metric space is sequentially compact.

3. Explain why the Least Upper Bound Property of $\mathbb{R}$ implies that every Cauchy sequence in $\mathbb{R}$ converges to some point of $\mathbb{R}$.

4. Prove that if X is a metric space, the each open ball $B(p, r)$, $r > 0$, $p \in X$ is an open subset of X.

5. Give an example of two metric space (X, d_X) and (Y, d_Y) and a continuous 1-1 onto function $F : X \to Y$ such that $F^{-1} : Y \to X$ is not continuous.

6. Let X = the space of continuous functions, $\mathbb{R}$-valued, on $[0,1]$ with metric $d_X(f, g) = \max\limits_{x \in [0,1]} |f(x) - g(x)|$.

   and Y = the space of continuous functions, $\mathbb{R}$-valued, on $[0,1]$ with metric $d_Y(f, g) = \left(\int_0^1 (f(x) - g(x))^2 \right)^{\frac{1}{2}}$

   Show that the identity function (f goes to f) from (X, d_X) to (Y, d_Y) is continuous.

7. In problem 6, show that the identity function from (Y, d_Y) to (X, d_X) is not continuous.

8 Suppose C is a subset of a metric space (X, d). Prove that if C is covering compact, C is closed in X <u>without</u> using covering compact $\Rightarrow$ (implies) sequentially compact, i.e., do this from the definition of covering compact.

9. (a) Prove that every sequence that converges must be a Cauchy sequence.

(b) Use part (a) to prove that if $C \subset X$, (X, d) a metric space, has the property that C as a metric space unto itself (with metric induced by d) is complete, then C must be closed in X.

10. Prove that a closed subset of a complete metric space is complete (in the induced metric).

11. Prove that if S is a subset of $\mathbb{R}^n$ and $S \subset \bigcup_{\lambda \in \Lambda} U_\lambda$, U_λ open in $\mathbb{R}^n$ for each $\lambda \in \Lambda$, then $\exists$ $\lambda_1, \lambda_2, \lambda_3 \ldots$ such that $S \subset \bigcup_{j=1}^{\infty} U_{\lambda_j}$ ("Every open cover has a countable subcover").

12. Outline the proof that every sequentially compact metric space has a countable dense subset.

13. Use problem 11 to show that every uncountable subset of $\mathbb{R}^n$ contains a condensation point. (i.e. S uncountable $\Rightarrow$ $\exists\, p \in S$ such that $B(p, \varepsilon) \cap S$ is uncountable for all $\varepsilon > 0$).

14. Define the Cantor set and prove that it is uncountable (you may use the Baire Category Theorem for the proof of uncountability)

15. Show that the (closed) unit ball around the 0-function in the metric space $C([0,1])$ is not compact (Here $C([0,1])$ = continuous $\mathbb{R}$-valued functions on $[0,1]$ with metric $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$).

16. If C is a closed subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, prove $\exists\, y_0 \in C$ such that $d(x,y_0) = \inf_{y \in C} d(x,y)$, i.e. y_0 is a closest point to x in C.

17. If S is a subset of a metric space X $(S \neq \phi)$, then show that the function $F: X \to \mathbb{R}$ defined by $F(x) = \inf_{y \in S} d(x,y)$ is a continuous function on X.

18(a) Suppose C_1, C_2 are compact subsets of a metric space X such that ~~[illegible]~~ $C_1 \cap C_2 = \phi$ (i.e. C_1 and C_2 are disjoint). Show $\exists\, \varepsilon > 0$ such that $d(x,y) \geq \varepsilon$ if $x \in C_1, y \in C_2$

(b) Given an example with $X = \mathbb{R}^2$ to show this statement in (the conclusion of) part (a) is false if C_1, C_2 are only assumed to be closed, not necessarily compact.

19. Given two metric spaces (X, d_X) and (Y, d_Y) define one of the reasonable metrics on $X \times Y$ and prove it is a metric space in your metric.

20. Prove that $X \times Y$ is sequentially compact if X and Y are sequentially compact (product space metric as in problem 19).

21. Outline how to do problems 19 & 20 for countably infinite products $\prod X_i$, (X_i, d_i) metric spaces.

22. Use the subsequence of subsequence of subsequence ... trick to prove:
If $f_i : [0,1] \to \mathbb{R}$ are a sequence of ~~functions~~ with $|f_i(x)| \le 1$, $\forall i = 1, 2, 3 \ldots$ $\forall x \in [0,1]$, then $\exists$ a subsequence f_{i_j} which has the property that for each rational $x \in [0,1]$, $f_{i_j}(x)$ converges (in $\mathbb{R}$).