Separable Metric Spaces & Subspaces
(Friday, May 1, 2009)

Definition: A metric space (X, d) is <u>separable</u> if there is a countable set $S \subset X$ such that S is dense in X.

Theorem: If (X, d_X) is separable and $Y \subset X$, then $(Y, d_X|_{Y \times Y})$ is separable. ($Y \neq \phi$ in what follows is assumed)

Slogan: "A subspace of a separable metric space is separable".

Idea that does <u>not</u> work: S countable dense in X $\Rightarrow$ $S \cap Y$ is countable dense in Y. The set $S \cap Y$ is countable all right, but it may not be dense! It could even be empty:

Example: $X = \mathbb{R}^2$, $Y = \{(a, b) \in \mathbb{R}^2 : a > 0, b = \sqrt{2}/a\}$
$S = \{(\alpha, \beta) : \alpha \in \mathbb{Q}, \beta \in \mathbb{Q}\}$.

Proof of Theorem: Let S be a countable dense subset of X. For each $s \in S$, $n \in \{1, 2, 3, 4 \ldots\}$, choose $y_{s,n} \in Y$ such that $d(s, y_{s,n}) \leq 1/n + \inf_{y \in Y} d(s, y)$. Note that such a $y_{s,n} \in Y$ exists by definition of inf (and $Y \neq \phi$).

Now $\{y_{s,n} : s \in S, n \in \{1, 2, 3 \ldots\}\}$ is countable since S is countable. We claim that $\{y_{s,n}\}$ is dense in Y.

To prove density, suppose $y_0 \in Y$, $\varepsilon > 0$ are given. We shall produce a y_{s_0,n_0} such that $d(y_0, y_{s_0,n_0}) < \varepsilon$, which suffices for density of $\{y_{s,n}\}$. For this, choose $n_0 \ni 1/n_0 < \varepsilon/100$ and $s_0 \ni d(y_0, s_0) < \varepsilon/100$, $s_0 \in S$.

Then $d(s_0, y_{s_0,n_0}) \le \frac{1}{n_0} + \inf\limits_{y\in Y} d(s_0, y)$

$< \frac{\varepsilon}{100} + \frac{\varepsilon}{100}$ because $\frac{1}{n_0} < \frac{\varepsilon}{100}$ & $\inf\limits_{y\in Y} d(s_0, y)$

$\le d(s_0, y_0) < \frac{\varepsilon}{100}$. So $d(s_0, y_{s_0,n_0}) < \frac{\varepsilon}{50}$.

Then $d(y_0, y_{s_0,n_0}) \le d(y_0, s_0) + d(s_0, y_{s_0,n_0})$

$$\le \frac{\varepsilon}{100} + \frac{\varepsilon}{50} < \varepsilon. \quad \square$$

Note that if we knew that for each $s \in S$, $\exists$ $y_s \in Y$ with $d(y_s, s) = \inf\limits_{y\in Y} d(s, y)$, then the proof could be simplified. But such a y_s may not exist. What we <u>are</u> sure exists is a $y_{s,n} \in Y$ with $d(y_{s,n}, s) \le \frac{1}{n} + \inf\limits_{y \in Y} d(s, y)$ & that turned out to be enough.

Knowing a metric space is separable is very useful because it means that every union $\bigcup_{\lambda \in \Lambda} U_\lambda$ of open sets in X is actually $= \bigcup_{j=1}^{+\infty} U_{\lambda_j}$ for some U_{λ_j}, $\lambda_j \in \Lambda$.

(Every cover of a set in X has a countable subcover). The proof of this follows a pattern that is by now very familiar! If S is a countable dense subset of X, and if U is an open subset of X, then U = union of all the open balls $B(s,r)$, $s \in S$, r positive rational number, that happen to be contained in U.
[Reason is as usual: If $p \in U$, $\exists\ \varepsilon > 0 \ni B(p,\varepsilon) \subset U$. Choose $r < \varepsilon/10$, r rational, $r > 0$ and $s \in S \ni d(p,s) < r$. Then $p \in B(s,r)$ and $B(s,r) \subset U$ because $q \in B(s,r) \Rightarrow d(s,q) < r \Rightarrow d(p,q) \le d(p,s) + d(s,q) < r + r \le 2\varepsilon/10 < \varepsilon$.]

So there is a set $\mathcal{B}$ of balls of $B(s,r)$ type such that $B \in \mathcal{B} \Rightarrow B \subset U_\lambda$, some λ & such that $\bigcup U_\lambda = \bigcup_{B \in \mathcal{B}} B$. Since the set of all balls of $B(s,r)$ type is countable, so is $\mathcal{B}$. Say $\mathcal{B} = B_1, B_2, B_3 \dots$ Choose $U_{\lambda_i} \supset B_i$. Then

$$\bigcup U_\lambda = U_{\lambda_1} \cup \dots \cup U_{\lambda_i} \cup \dots \qquad \square$$