Notes for Friday, April 17, 2009: Heine-Borel and All That

Since we know that a metric space (X, d) is (covering) compact if and only if it is sequentially compact, we can check compactness in any given instance in two different ways.

Example:

(1) $[0, 1]$

(a) sequentially compact: If $\{x_i\}$ is a sequence in $[0, 1]$, then there is a convergent subsequence by (i) every bounded sequence has a convergent subsequence because $\text{l.u.b.}\{\alpha : x_i \le \alpha$ for only finitely many i values$\}$ is a limit of some subsequence (cf. completeness of $\mathbb{R}$ proof earlier).
or (ii) subdivision argument: $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ contains x_i for infinitely many i values.
Let I_1 = one of the two that does have ∞ly many x_i.
Let I_2 = a half of I_1 that contains x_i, ∞ly many i.
So $\ldots I_n \supseteq I_{n+1} \ldots$ and length $I_n = 1/2^n$.
Then lub of lefthand endpoints of I_n's $(\underset{(only)}{=}$ glb of right hand endpoints$)$ is a limit of a sequence of x_i's.
Since $[0, 1]$ is closed, subsequence limit $\in [0, 1]$ ∎

(b) Covering compact: U_λ, $\lambda \in \Lambda$ open cover.
Let $S = \{\alpha \in [0, 1] : [0, \alpha] \subset$ some union of finitely many U_λ's$\}$. Then $0 \in S$.

Let α_0 = sup of such α's. (Note that $\alpha \in S$ $\Rightarrow$ if $\beta < \alpha$, then $\beta \in S$). If $\alpha_0 < 1$, then since $\alpha_0 \in U_{\lambda_0}$ for some λ_0, it follows that $\alpha_0 + \varepsilon$, ε small $\in S$, a contradiction. If $\alpha_0 = 1$, then $\alpha_0 = 1 \in U_{\lambda_0}$ while $\alpha - \varepsilon \in S$ every $\varepsilon > 0$, so $[0, 1]$ is contained in a finite union of U_λ's, namely U_{λ_0} together with finitely many U_λ's that cover $[0, 1-\varepsilon]$ where $\varepsilon > 0$ is so small that $(1-\varepsilon, 1] \subset U_{\lambda_0}$. □

Then $[-M, M] \times \dots \times [-M, M] \subset \mathbb{R}^n$ is compact for every $M \in \mathbb{R}$. (Reason: You proved in homework that product — finite or countably infinite — of sequentially compact spaces is sequentially compact.

(Heine Borel)
Cor: A closed bounded set in $\mathbb{R}^n$ is compact.
Proof C closed bounded $\Rightarrow$ C is a closed subset of $[-M, M] \times \dots \times [-M, M]$, some $M > 0$. If U_λ $\lambda \in \Lambda$ is an (open) cover of C then U_λ, $\lambda \in \Lambda$ together with $\mathbb{R}^n - C$ (which is open) cover $[-M, M] \times \dots \times [-M, M]$, say $U_{\lambda_1}, \dots, U_{\lambda_k}$ together with $\mathbb{R}^n - C$. But then $C \subset U_{\lambda_1} \cup \dots \cup U_{\lambda_k}$ (since $\mathbb{R}^n - C$ contains no points of C!) □

Lemma: $C \subset \mathbb{R}^n$ compact $\Rightarrow$ C closed and bounded.

Sequence view: If C were unbounded, then there would be a sequence $x_i \in C$ with $d(0, x_i) \to +\infty$ as $i \to \infty$. This would have no convergent subsequence in $\mathbb{R}^n$ and hence certainly no convergent subsequence with limit in C. If p is adherent to C but not in C, then $\exists$ a sequence $x_i \in C$ with limit $x_i = p$. C compact $\Rightarrow$ $\exists$ subsequence which converges to a point of C. Hence $p \in C$. $\square$

Covering view: $C \subset \bigcup_{n=1}^{+\infty} \{x \in C : d(x, \vec{0}) < n\}$ (any C in $\mathbb{R}^n$!)

So C compact $\Rightarrow$ $C \subset \bigcup_{n=1}^{N} \{x \in C : d(x, \vec{0}) < n\}$

$= \{x \in C : d(x, \vec{0}) < N\}$, so C is bounded.

If x_0 is in $\mathbb{R}^n$ but not in C, then
$$C \subset \bigcup_{n=1}^{+\infty} \{x \in C : d(x_0, x) > \tfrac{1}{n}\}.$$

So $C \subset \bigcup_{n=1}^{N} \{x \in C : d(x_0, x) > \frac{1}{n}\}$, some N

So
$$C \subset \{x \in C : d(x_0, x) > \tfrac{1}{N}\}.$$ Hence $C \cap \{x \in C : d(x, x_0)\} < \frac{1}{N}$ is empty. (So x_0 is not adherent to C) Hence C has no adherent points that are not in C and C is closed. $\square$

Thus we have the full Heine-Borel Theorem:

Theorem (Heine-Borel): A set $C \subset \mathbb{R}^n$ is compact if and only if C is closed and bounded.

Note that one direction of implication works in arbitrary metric spaces: If (X, d) is a metric space and $C \subset X$ is compact, then C is closed in X and bounded (in the sense that for each $p_0 \in X$, $\exists\, M_{p_0} \ni C \subset \{x \in X : d(p_0, x) < M_{p_0}\}$). In particular, if the whole space X is compact, it is bounded (same proof as before for $\mathbb{R}^n$ case).

Exercise: Also, a compact set is always complete (which implies closed) — this is stronger than closed sometimes!

But a set can be complete and bounded in a metric space X without being compact: the closed unit ball (around 0) in the space of all continuous functions on $[0,1]$ with $\mathbb{R}$ values.

$$d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

[$C([0,1])$ is complete so this closed (bounded) subset is complete]

The condition, in addition to completeness, to guarantee compactness is being "totally bounded" (not just bounded). See G&G text for details of that situation