Compactness implies and Conversely Sequential Compactness

(Monday, April 13, 2009) & Wednesday, April 15, 2009)

Def.
A metric space (X,d) is (covering) **compact** if, for every collection U_λ, $\lambda \in \Lambda$, of open sets with $\bigcup_{\lambda\in\Lambda} U_\lambda = X$, there is a finite subcollection $U_{\lambda_1}, \ldots, U_{\lambda_k}$ with $\bigcup_{j=1}^{k} U_{\lambda_j} = X$. Slogan: Every open cover has a finite subcover.

A metric space (X,d) is **sequentially compact** if every sequence $\{x_i\}$ in X has a subsequence $\{x_{i_j} : j=1,2..\}$ that converges. Slogan: Every sequence has a convergent subsequence.

Theorem: A metric space is compact if and only if it is sequentially compact.

We prove here compact $\Rightarrow$ sequentially compact. The $\Leftarrow$ implication will be done later

Proof that compact $\Rightarrow$ sequentially compact: Suppose that X is (covering) compact and that $\{x_i\}$ is a sequence in X which has no convergent subsequence. To obtain a contradiction we shall use this lemma:

Lemma: If $\{x_i\}$ is a sequence in a metric space X, if $p \in X$, and if no subsequence of $\{x_i\}$ converges to p, then $\exists\, \varepsilon > 0$ such that $x_i \in B(p,\varepsilon)$ for only finitely many values of i.

Proof: If no such $\varepsilon > 0$ exists, then $x_i \in B(p,1)$ for infinitely many i, $x_i \in B(p,\frac{1}{2})$ for infinitely many i, $x_i \in B(p,\frac{1}{3})$ for infinitely many i. Then $\exists$ a subsequence $x_{i_1}, x_{i_2}, x_{i_3} \ldots$ of $\{x_i\}$ such that $x_{i_j} \in B(p, 1/j)$ for all $j = 1,2,3$.
So $\{x_{i_j}\}$ converges to p, which was assumed not to happen. $\square$

Returning to $\{x_i\}$ with no convergent subsequence, $\forall p \in X$ $\exists$ (by the Lemma) $\varepsilon_p > 0$ $\ni$ $x_i \in B(p,\varepsilon_p)$ for only finitely many i. Then $B(p,\varepsilon_p)$, $p \in X$ is an open cover of X.

So $\exists$ a finite subcover $B(p_1, \varepsilon_{p_1}), \ldots, B(p_k, \varepsilon_{p_k})$
i.e. $B(p_1, \varepsilon_{p_1}) \cup B(p_2, \varepsilon_{p_2}) \cup \cdots \cup B(p_k, \varepsilon_{p_k}) = X$.
But $x_i \in B(p_j, \varepsilon_{p_j})$ for only finitely many i, for each $j = 1, \ldots, k$. This clearly contradicts the fact that there are infinitely many i values possible: $i = 1, 2, 3 \ldots$ $\square$

End of April 13

Here is an alternative argument for sequential compactness being implied by compactness:
Let $\{x_i\}$ be a sequence in the compact metric space X. If $\{x_i\}$ contains any point x_0 infinitely often, it obviously has a subsequence converging to x_0. So if $\{x_i\}$ had no convergent subsequence, no point occurs infinitely often. Eliminating the finite repetitions, we get a sequence $\{y_j\}$ without repetitions and again this has no convergent subsequence. It follows that the set $C_k = \{y_j : j \geq k\}$ is closed (in X) since it contains all its sequential limits. Set $U_k = X - C_k$. Then each U_k is open. Moreover

$$X - \bigcup_{k=1}^{+\infty} U_k = \bigcap_{k=1}^{+\infty} (X - (X - C_k)) = \bigcap_{k=1}^{+\infty} C_k = \phi.$$

Thus $\bigcup_{k=1}^{+\infty} U_k = X$. Hence $\bigcup_{k=1}^{N} U_k = X$ for some N. But $X - \bigcup_{k=1}^{N} U_k = \bigcap_1^N C_k \neq \phi$ since $\bigcap_1^N C_k = C_N \neq \phi$ $\square$.

Note we needed only: Every countable cover has a finite subcover here!

In particular, we have proved:

Theorem: If (X,d) has the property that every countable covering U_j, $j=1,2,3\ldots$ has a countable subcover (i.e. $\exists N \ni \bigcup_{j=1}^{N} U_j = X$) then (X,d) is sequentially compact.

It is not hard to prove the converse:

Theorem: If X is sequentially compact, then every countable cover has a finite subcover.

Proof: Suppose $X - \bigcup_{j=1}^{N} U_j \neq \phi$ for all $N = 1,2,3\ldots$

Choose $x_N \in X - \bigcup_{j=1}^{N} U_j$ for each $N = 1,2,3,\ldots$

By hypothesis, $\exists$ x_{N_k}, $k=1,2,3\ldots$ a subsequence such that $x_{N_k} \to x_0$. Now $x_0 \in U_{N_0}$ for some N_0. Hence for $k \geq k_0$ (k sufficiently large) $x_{N_k} \in U_{N_0}$ (since $x_{N_k} \to x_0$ and x_0 is in the open set U_{N_0}). In particular, there is an N_k, some k, with $N_k > N_0$ and $x_{N_k} \in U_{N_0}$. Thus $x_{N_k} \in \bigcup_{j=1}^{N_0} U_j \subset \bigcup_{j=1}^{N_k} U_j$. This contradicts the choice of the x_N's, x_{N_k} in particular. □

Note all we need to do to show (X,d) sequentially compact $\Rightarrow$ (X,d) is compact is prove:

Proposition: If (X,d) is sequentially compact, then every open cover of (X,d) has a countable subcover.

We do this by an argument similar to that we used (in the homework assignment) to show that for any set in $\mathbb{R}^n$, every open cover has a countable subcover, using rational balls. We just need some idea like rational balls.

Definition: A subset $S \subset X$ is <u>dense</u> in X if the closure of S in $X = X$.

Proposition: If (X,d) is sequentially compact, then there is a countable subset S of X which is dense in X.

Assuming this proposition for a moment, we complete the proof that (X,d) sequentially compact $\Rightarrow (X,d)$ compact using the proposition.

Proposition: If a metric space (X,d) has a countable dense subset, then every open cover of X has a countable subcover.

Proof: Let $\mathcal{U}$ = the set of balls $B(p,r)$ where $r>0$ is rational and $p \in S$ (S = the countable dense subset). If $U \subset X$ is open and $q \in U$, then $\exists$ a ball $B \in \mathcal{U}$ with $q \in B$ and $B \subset U$. (proof: $B(q,\varepsilon) \subset U$ for some $\varepsilon > 0$. Choose $r < \varepsilon/10$, $r>0$, and $p \in S \ni d(p,q) < r$. Then $q \in B(p,r)$ and $x \in B(p,r) \Rightarrow d(q,x) \le d(x,p) + d(p,q) < 2r < \varepsilon$ so $x \in U$. $\square$). So every open set U = union of all the balls $\in \mathcal{U}$ that are contained in U. If U_λ $\lambda \in \Lambda$ is a covering of X by open sets, let $\mathcal{U}_0$ = all the balls B in $\mathcal{U}$ such that $B \subset U_\lambda$, some λ. Then $X = \bigcup U_\lambda$ = union of all the balls in $\mathcal{U}_0$. But $\mathcal{U}_0$ is countable, say $B(p_j, r_j)$, $p_j \in S$, r_j rational. Let $U_{\lambda_j} \supset B(p_j, r_j)$. Then $\bigcup_{j=1}^{\infty} U_{\lambda_j} = X$

$\square$

It remains to show that every sequentially compact metric space contains a countable dense subset. For this, we make a definition:

Def.: An ε-net in a metric space (X, d) is a set S such that $d(p, q) \geq \varepsilon$ for $\forall\, p, q \in S \ni p \neq q$ and for $\forall\, x \in X$, there is a point $p_x \in S$ such that $d(x, p_x) < \varepsilon$.

Lemma: If (X, d) is sequentially compact, then for each $\varepsilon > 0$, there is a finite set S_ε which is an ε-net.

Proof: Given $\varepsilon > 0$, choose $x_1 \in X$. If $B(x_1, \varepsilon) = X$, we are done: $S_\varepsilon = \{x_1\}$ will do the job. If $B(x_1, \varepsilon) \neq X$, then choose $x_2 \in X - B(x_1, \varepsilon)$. If $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) = X$, we are done. If not, choose $x_3 \in X - (B(x_1, \varepsilon) \cup B(x_2, \varepsilon))$. Continue in this way. The process must stop (with $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots \cup B(x_k, \varepsilon) = X$, some k) because otherwise $x_1, x_2, x_3, x_4 \ldots$ would be an infinite sequence with no convergent subsequence. $\square$

Now if (X, d) is sequentially compact, set $S = S_1 \cup S_{1/2} \cup S_{1/3} \cup \ldots$ where S_ε is a finite ε-net as above. Since S is a countable union of finite sets, S is countable. And clearly, S is dense in X since for each $k = 1, 2, 3 \ldots$, (given $p_0 \in X$), there is an $x \in S_{1/k}$ with $d(x, p_0) \leq 1/k$. $\square$

Thus the proof that sequential compactness $\Rightarrow$ compactness is at last complete.