Notes on Completeness of the Real Numbers $\mathbb{R}$ from Least Upper Bound Property

Def: M is an upper bound for a set $S \subset \mathbb{R}$ if $s \leq M$ for all $s \in S$.
M_0 is a least upper bound for a set $S \subset \mathbb{R}$ if M_0 is an upper bound for S and, for every $M \in \mathbb{R}$ that is an upper bound for S, $M_0 \leq M$.

Least Upper Bound Property for $\mathbb{R}$: If $S (\subset \mathbb{R})$ has an upper bound, it has a least upper bound.

Now suppose $\{x_i\}$ is a Cauchy sequence in $\mathbb{R}$. Then $\exists M \in \mathbb{R} \ni |x_i| \leq M$ for all $i = 1, 2, 3, \ldots$
(Reason: $\exists i_0 \ni |x_{i_0} - x_j| \leq 1$ for all $j \geq i_0$ by definition of Cauchy sequence. Set $M = \max(|x_1|, |x_2|, \ldots |x_{i_0-1}|, |x_{i_0}|+1)$. Then M is an upper bound for $|x_i|$, all i.)

Now let $S = \{\lambda \in \mathbb{R} : x_j \leq \lambda$ for only finitely many j values$\}$

Then (1) S is not empty since $-M-1 \in S$

(2) If $\lambda_0 \in S$, then $\lambda \in S$ if $\lambda < \lambda_0$

(3) S is bounded above (has an upper bound) since $M+1 \notin S$

In (1) and (3), M is an upper bound for $\{|x_i| : i = 1, 2, 3, \}$ as indicated/chosen above.

Let α = the least upper bound of S (l.u.b. S).

Then, for each $\varepsilon > 0$:

(a) $\alpha - \varepsilon/3 \in S$ (Reason: If not, if $\alpha - \varepsilon/3 \notin S$, then by property (2), no element λ_0 of S is $> \alpha - \varepsilon/3$. Hence α cannot be the least upper bound of S since $\alpha - \varepsilon/3$ is an up. bd.

(b) $\alpha + \varepsilon/3 \notin S$: this is because α is an upper bound for S.

Now (a) and (b) together imply that, for ~~only~~ infinitely many j, $x_j \in (\alpha - \varepsilon, \alpha + \varepsilon)$ since there are infinitely many j with $x_j \le \alpha + \varepsilon/3$ but only finitely many with $x_j \le \alpha - \varepsilon/3$. Since this is true for each $\varepsilon > 0$, we can find a subsequence of $\{x_i\}$ converging to α as follows: Choose $x_{i_1} \in (\alpha - 1, \alpha + 1)$, $x_{i_2} \in (\alpha - \frac{1}{2}, \alpha + \frac{1}{2})$ and $i_2 > i_1$, $x_{i_3} \in (\alpha - \frac{1}{3}, \alpha + \frac{1}{3})$ and $i_3 > i_2$, etc. [the choice is always possible since each $(\alpha - \frac{1}{n}, \alpha + \frac{1}{n})$ contains x_j for infinitely many j values so we can always find x_{i_n} with i_n as large as desired, in particular $i_n > i_{n-1}$ and $x_{i_n} \in (\alpha - \frac{1}{n}, \alpha + \frac{1}{n})$]

We have already seen that if a Cauchy sequence $\{x_i\}$ has a subsequence converging to α, then the whole sequence $\{x_i\}$ converges to α. Hence $\{x_i\}$ converges to α in our case. The proof is complete that every Cauchy sequence in $\mathbb{R}$ has to converge to a point of $\mathbb{R}$ $\square$

Note that in the process we proved: Every sequence $\{x_i\}$ in $\mathbb{R}$ such that there is an $M \in \mathbb{R} \ni |x_i| \le M$, all i (we say the sequence is "bounded") has a convergent subsequence. Slogan: "Every bounded sequence in $\mathbb{R}$ has a convergent subsequence."

This is important!