Summary Notes for Friday, April 3 (four pages)

(X, d) metric space.

Open set (definition) U <u>open</u> $\iff \forall p \in U, \exists \varepsilon_p > 0 \ni B(p, \varepsilon_p) \subset U$.

Here $B(p, r) \overset{\text{def.}}{=} \{x \in X : d(x, p) < r\}$

Closed set: Definition: C <u>closed</u> $\iff$ Every point adherent to C belongs to C.

This is equivalent to: $X - C$ is open.
So one could use $X - C$ being open as the definition of being closed. Equivalence is in book (1.8, p.6)

Important properties of open sets
(1) $\emptyset, X$ are open (2) Each $U_\lambda, \lambda \in \Lambda$ open $\Rightarrow \bigcup_{\lambda \in \Lambda} U_\lambda$ is open

(3) If $U_1, \ldots, U_N$ are open, then $\bigcap_{j=1}^{N} U_j$ is open.

Proofs: (1) is clear by directly checking the definition
For (2), suppose $p \in \bigcup U_\lambda$. Then $p \in U_{\lambda_p}$ for some $\lambda_p \in \Lambda$ by definition of union of sets. So $\exists\, \varepsilon > 0$ such that $B(p, \varepsilon) \subset U_{\lambda_p}$ (because U_{λ_p} is open). Then $B(p, \varepsilon) \subset \bigcup_{\lambda \in \Lambda} U_\lambda$

For (3), suppose $p \in \bigcap_{j=1}^{N} U_j$. Then for each $j = 1, \ldots, N$, there exists $\varepsilon_j > 0$ such that $B(p, \varepsilon_j) \subset U_j$: this is because $p \in U_j$ and U_j is open. Set $\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_N)$. Then $\varepsilon > 0$ and $B(p, \varepsilon) \subset U_j$ for all $j \in \{1, \ldots, N\}$ because $B(p, \varepsilon) \subseteq B(p, \varepsilon_j) \subset U_j$ (since $\varepsilon \le \varepsilon_j$). So $B(p, \varepsilon) \subset \bigcap_{j=1}^{N} U_j$. $\square$

Note that finiteness is needed for property (3) to hold in general: $\bigcap_{j=1}^{+\infty} \left(-\frac{1}{j}, \frac{1}{j}\right) = \{0\}$, and $\{0\}$ is not open in $\mathbb{R}$ (with usual $d(x, y) = |x - y|$) even though $\left(-\frac{1}{j}, \frac{1}{j}\right) = B(0, \frac{1}{j})$ is open for each $j = 1, 2, 3 \ldots$

② Since C closed $\iff X \setminus C$ open, properties (1), (2) & (3) give rise to properties of closed sets by taking complements. For this, recall de Morgan's Laws

(i) complement of a union = intersection of complements

$$X \setminus \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right) = \bigcap_{\lambda \in \Lambda} (X \setminus A_\lambda)$$

(ii) complement of intersection = union of complements

$$X \setminus \left(\bigcap_{\lambda \in \Lambda} A_\lambda \right) = \bigcup_{\lambda \in \Lambda} (X \setminus A_\lambda)$$

Thus (1) ϕ open $\Rightarrow X = X \setminus \phi$ closed

X open $\Rightarrow \phi = X \setminus X$ closed

and (using deMorgan's Laws):

(2) C_λ, $\lambda \in \Lambda$ closed $\Rightarrow \bigcap_{\lambda \in \Lambda} C_\lambda$ is closed

(3) $C_1, \ldots, C_N$ closed $\Rightarrow \bigcup_{j=1}^{N} C_j$ is closed.

Check of (3), for example: $\bigcup_{j=1}^{N} C_j$ is closed if $X \setminus \left(\bigcup_{j=1}^{N} C_j \right)$ is open. But

$$X \setminus \left(\bigcup_{j=1}^{N} C_j \right) = \bigcap_{j=1}^{N} (X \setminus C_j) \quad \text{(deMorgan).}$$

Now each $X \setminus C_j$ is open since each C_j is closed.

By property (3) for open sets (finite intersection of open sets is open), $\bigcap_{j=1}^{N} (X \setminus C_j)$ is open.

Hence $\bigcup_{j=1}^{N} C_j$ is closed, as required. □

Instructive exercise: Check properties (1), (2), (3) for closed sets directly, using the "adherent-points are in the set" definition.

③ Convergence of sequences

Recall definition: A sequence $\{x_i : i = 1,2,3..\}$ of points in a metric space X converges to $x_0 \in X$ if $\forall \varepsilon > 0$, $\exists N_\varepsilon \ni n > N_\varepsilon \Rightarrow d(x_0, x_n) < \varepsilon$.

This is the same as saying $n > N_\varepsilon \Rightarrow x_n \in B(x_0, \varepsilon)$, which is open

So this suggests convergence of sequences can be expressed in terms of open sets. This is true:

A sequence $\{x_i\}$ converges to x_0 if and only if $\forall$ open set U with $x_0 \in U$, $\exists N_U \ni n > N_U \Rightarrow x_n \in U$.

Proof of equivalence: convergence $\Rightarrow$ N_U condition since: if $x_0 \in U$, U open, $\exists \varepsilon > 0$ with $B(x_0, \varepsilon) \subset U$. We can take $N_U = N_\varepsilon$. The N_U condition $\Rightarrow$ convergence is clear since for each $\varepsilon > 0$, we can take $U = B(x_0, \varepsilon)$. and then $N_\varepsilon = N_U$. □

Characterization of closed sets in terms of sequence limits.

A set C is closed $\Longleftrightarrow$ for every sequence $\{x_i \in C\}$ which converges to some point $x_0 \in X$, $x_0 \in C$.

Slogan: "C is closed $\Leftrightarrow$ C contains all its sequential limits"

Proof: $\Rightarrow$: Suppose $\{x_i \in C\}$ converges to $x_0 \in X$. Then x_0 is adherent to C because:, given $\varepsilon > 0$, $x_i \in B(x_0, \varepsilon)$ if $i > N_\varepsilon$; in particular, $x_i \in B(x_0, \varepsilon)$ for some i, and $x_i \in C$ by hypothesis. Since x_0 is adherent to C, $x_0 \in C$ if C is closed.

$\Leftarrow$: Suppose p is adherent to C. Then $\exists\, x_1 \in C \cap B(p, 1)$, $x_2 \in C \cap B(p, \frac{1}{2})$, $x_3 \in C \cap B(p, \frac{1}{3})$, etc. The sequence $\{x_1, x_2, x_3 ..\}$ converges to x_0. So, if

(4) C contains all its sequential limits, then $x_0 \in C$.
Thus every point adherent to C is in C and C is closed ∎

Note that knowing the open sets of X tells one which sequences converge and to which point. And what we just proved shows that knowing which sequences converge and to which point tells one which sets in X are closed and hence which sets are open (since open $\Leftrightarrow$ complement is closed). So knowing about convergence of sequences is the same as knowing which sets are open.

Cauchy sequences and completeness:

Def: A sequence $\{x_i \in X : i = 1, 2, 3 \ldots\}$ is a Cauchy sequence
$\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \ni i, j > N_\varepsilon \Rightarrow d(x_i, x_j) < \varepsilon$.

A metric space X is complete if every Cauchy sequence in X converges to some point in X.

Examples: (1) Any set X with the "0, 1" metric (all distances = 0 or 1)

(2) real numbers $\mathbb{R}$, $d(x, y) = |x - y|$ (reason will be discussed next time)

Nonexamples: (1) $B((0,0), 1)$ $(= \{(x_1, x_2) : x_1^2 + x_2^2 < 1\} \in \mathbb{R}^2)$

(2) $\mathbb{R}^2 - \{(0,0)\}$

(3) $(0, 1)$

General principle: A non-complete metric space always looks like (is) the result of taking a complete metric space and removing some points (prob. 7, p. 13)