Solutions for Midterm I (April 27, 2009)

1. Prove that if X is a metric space, then each open ball B(p,r), r>0, p ∈ X is an open subset of X.

Need to Check: $\forall\, q \in B(p,r)$ $\quad \exists\, \varepsilon_q > 0 \ni B(q,\varepsilon_q) \subset B(p,\varepsilon)$.

For this:

Set $\quad \varepsilon_q = \varepsilon - \mathrm{dis}(p,q) > 0$ since $d(p,q) < \varepsilon$.

Then $\quad x \in B(q,\varepsilon_q) \Longleftrightarrow d(q,x) < \varepsilon - d(p,q)$.

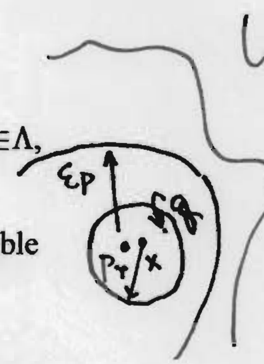
So if $x \in B(q,\varepsilon_q)$, then

$$d(p,x) \le \mathrm{dis}(x,q) + d(q,p)$$

$$< \varepsilon - d(p,q) + d(p,q) \le \varepsilon.$$

So $\quad x \in B(p,\varepsilon)$. $\square$

2. Prove that if S is a subset of $\mathbb{R}^n$ and $S \subset \bigcup_{\lambda\in\Lambda} U_\lambda$, U_λ open in $\mathbb{R}^n$ for each $\lambda\in\Lambda$, then $\exists$ $\lambda_1, \lambda_2, \lambda_3$ …such that $S \subset \bigcup_{j=1}^{\infty} U_{\lambda_j}$ ("Every open cover has a countable subcover").



Let $\mathcal{S}$ = set of "rational balls" $B((x_1, \dots, x_n), r)$ where x_i are rational, $r > 0$ is rational.

Any open set $U \subset \mathbb{R}^n$ = union of the rational balls $\subset U$

(Reason: $p \in U \Rightarrow B(p, \varepsilon_p) \subset U$ $\varepsilon_p > 0$; choose r rat., $0 < r < \varepsilon/10$

Choose rat pt. $\vec{x} \ni d(p, \vec{x}) < r$

Then $p \in B(\vec{x}, r)$ & $B(\vec{x}, r) \subset U$ because $q \in B(\vec{x}, r)$

$\Rightarrow d(p,q) \leq d(p,x) + d(x,q) < 2r < \varepsilon \Rightarrow \vec{q} \in B(p, \varepsilon_p) \subset U$).

Let $\mathcal{S}'$ = set of rational balls $B \ni B \subset U_\lambda$, some λ. $\mathcal{S}'$ is countable (since $\mathcal{S}$ is countable)

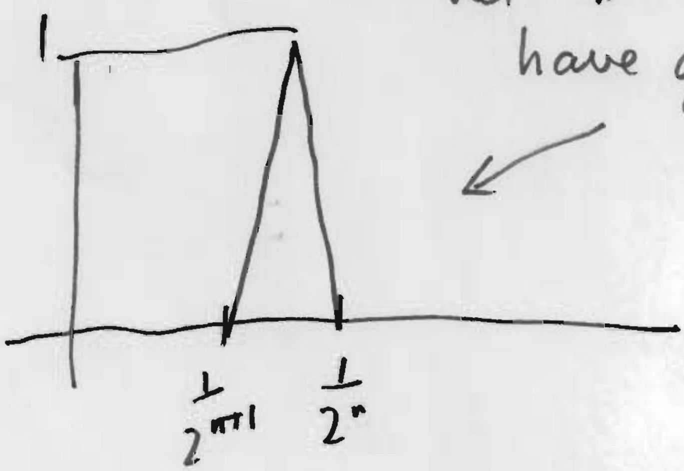
say $\mathcal{S}' = B_1, B_2, B_3 \dots$

choose $U_{\lambda_j} \ni B_j \subset U_{\lambda_j}$. Then

$$\bigcup_{\lambda\in\Lambda} U_\lambda = \text{union of } B,\ B \in \mathcal{S}' \subseteq \bigcup_{j=1}^{+\infty} U_{\lambda_j} \quad \square$$

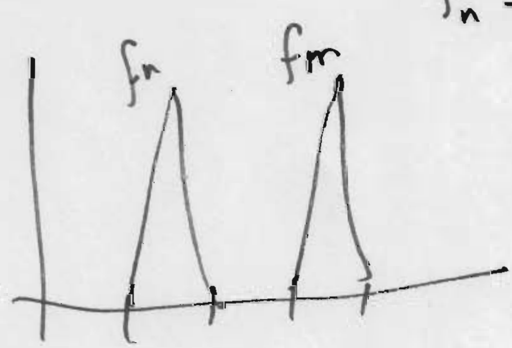
3. Show that the (closed) unit ball around the 0-function in the metric space C([0,1]) is not compact (Here C([0,1]) = continuous ℝ-valued functions on [0,1] with metric $d(f,g) = \max_{x\in[0,1]}|f(x)-g(x)|$).

Let $f_n(x)$ $n = 1, 2, 3 \cdots$ have graph as shown

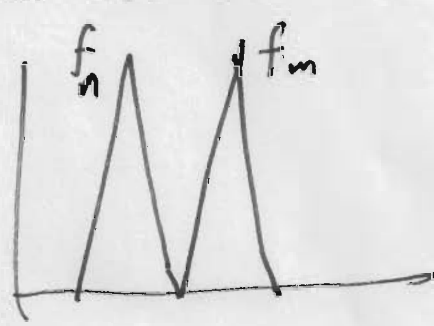


$d(0, f_n) = 1$

$d(f_n, f_m) \geq 1$ since $f_n = 1$ where $f_m = 0$:



or



So $\{f_n\}$ has no convergent subsequence.

4. Prove that every sequentially compact metric space has a countable dense subset.

It suffices to show for each $\varepsilon>0$ $\exists$ a finite set $S_\varepsilon \ni \forall x \in X$, $\exists y \in S_\varepsilon \ni$ $d(x,y)<\varepsilon$. For then $\bigcup_{j=1}^{+\infty} S_{1/j}$ is countable (countable union of finite sets) and dense.

Fix $\varepsilon>0$. Choose $x_1 \in X$. If $B(x_1,\varepsilon)=X$, done. If not, choose $x_2 \in X - B(x_1,\varepsilon)$, i.e. $d(x_1,x_2)\geq\varepsilon$. If $B(x_1,\varepsilon)\cup B(x_2,\varepsilon)=X$, done. If not, choose $x_3 \in X-(B(x_1,\varepsilon)\cup B(x_2,\varepsilon))$ i.e., $d(x_1,x_3)\geq\varepsilon$ $d(x_2,x_3)\geq\varepsilon$.

Continue. Process must terminate at finite stage since otherwise $x_1, x_2, x_3 \ldots$ is an infinite sequence of points each with distance $\geq\varepsilon>0$ to all others. This has no convergent subsequence hence cannot occur, since X is seq. compact $\square$

5. Suppose (X,d) is a complete metric space and $\overline{B}(p_i, r_i)$ $(\stackrel{\text{def.}}{=}\{x : d(x,p_i) \le r_i\})$ is a sequence of closed balls with $\lim_{i\to\infty} r_i = 0$ and $\overline{B}(p_{i+1}, r_{i+1}) \subset \overline{B}(p_i, r_i)$ for each $i = 1.2.3\ldots$

Prove that $\exists$ $p_0 \in X$ with $p_0 \in \bigcap_{i=1}^{\infty} \overline{B}(p_i, r_i)$.

Note that $i \ge j \Longrightarrow d(p_i, p_j) \le r_j$ since $p_i \in \overline{B}(p_i, r_i) \subset \overline{B}(p_j, r_j)$

Since $\lim r_j = 0$, this implies $\{p_j\}$ is a Cauchy sequence (detail: Given $\varepsilon > 0$, choose $j_0 \ni i \ge j_0 \Rightarrow r_i < \varepsilon$.

Then $i, j \ge j_0 \Rightarrow d(p_i, p_j) \le r_{\min(i,j)} < \varepsilon$.)

The sequence $\{p_j\}$ hence converges to, say, p_0.

Then $p_0 \in \overline{B}(p_i, r_i)$ for each i

since $p_j \in \overline{B}(p_i, r_i)$ for all $j \ge i$.

and $\overline{B}(p_i, r_i)$ is closed. So

$p_0 \in \bigcap_{i=1}^{+\infty} \overline{B}(p_i, r_i)$ $\square$