A Quick Tour of Basic Ideas of Algebraic Topology
(Friday, May 22 & Wednesday, May 27, 2009)

Basic extension question: $E \subset X$ (X normal top. space. or ~~just~~ a metric space is general enough to illustrate the ideas), E closed, $f: E \to Y$ continuous, $\exists$? $\hat{f}: X \to Y$ such that $\hat{f}(x) = f(x)$ for all $x \in E$?

Tietze Extension: Yes, if $Y = [0,1]$ or $Y = \mathbb{R}$.

But in general, the answer can be no.

Fundamental example: $X = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$
$E = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$, $Y = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$,
$f: E \to Y$ is the identity map $f((x,y)) = (x,y)$.
X, Y, E, f as indicated:
Theorem: f does not "extend to X", i.e., there is no continuous function $\hat{f}: X \to Y$ such that $\hat{f}(p) = f(p)$ $(= p)$ for all $p \in E$.

This theorem is not "obvious" though it is intuitively appealing. To prove it (which we shall do only in outline), we need the idea of "winding number". For this, suppose $F: S^1 \to S^1$ is a continuous function, $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$. Define a mapping $[0,1] \to S^1$ by $t \to$ "$F(e^{2\pi i t})$"
($e^{2\pi i t} = \cos 2\pi t + i \sin 2\pi t$) or $t \to (F((\cos 2\pi t, \sin 2\pi t))$

This is a closed curve: 0 and 1 go to the same point of S^1, namely $F((1,0))$.
If we chose θ_0 such that $(\cos\theta_0, \sin\theta_0) = F((1,0))$, then it is not hard to see that that there is one and only one function $t \to \theta_t$, $t \in [0,1]$, with the specified value of θ_0 and with θ_t a continuous function of t and with
$$(\cos\theta_t, \sin\theta_t) = F((\cos 2\pi t, \sin 2\pi t))$$
for all $t \in [0,1]$. The function θ_t is a polar coordinate angle for $F((\cos 2\pi t, \sin 2\pi t))$.

Since $F((\cos 2\pi(1), \sin 2\pi(1)) = F((\cos 2\pi(0), \sin 2\pi(0))$ $(= F((1,0))$, it must be that
$$(\cos\theta_1, \sin\theta_1) = (\cos\theta_0, \sin\theta_0).$$
So $\theta_1 - \theta_0$ is an integer multiple of 2π,

Definition: The winding number of $F: S^1 \to S^1$

$$= \frac{1}{2\pi}(\theta_1 - \theta_0). \quad \text{(This is an integer!)}$$

It is easy to check that this is independent of the choice of the "initial polar coordinate angle" θ_0.

Since the winding number is an integer which intuitively depends in a continuous way on F, it is natural to suppose that the winding number cannot change when F is continuously "deformed". This intuition has the following precise meaning:

Theorem ("homotopy invariance"): If $H: S^1 \times [0,1] \to S^1$ is a continuous function, ~~and~~ then the winding number of $H(\cdot\,, s) : S^1 \to S^1$ is independent of s.

Proof idea: wind no of $H(\cdot, s)$ is continuous in s, but integer valued $\square$

Assuming this proof idea works (it does), we get as a corollary:

Theorem ("no retraction of B^2 onto S^1"): There is no continuous function $F: \{(x,y): x^2+y^2 \leq 1\} \to \{(x,y): x^2+y^2=1\}$ such that $F((x,y)) = (x,y)$ for all (x,y) with $x^2+y^2=1$.

Proof: Suppose such an F existed. Then define $H: S^1 \times [0,1] \to S^1$ by $H((x,y),s) = F(((1-s)x, (1-s)y))$. Then $H(\cdot\,,0)$ is the identity map of $S^1 \to S^1$ while $H(\cdot\,,1)$ is a constant map with value $F((0,0))$. The winding number of the identity is 1 (by definition tracing) while that of a constant map is obviously 0. This contradicts the "homotopy invariance" theorem above.

$\square$

It is natural to ask if something similar works for $S^n = \{(x_1, \ldots, x_{n+1}) : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\} \subset$

$B^{n+1} = \{(x_1, \ldots, x_{n+1}) : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 \leq 1\} \quad (\subset \mathbb{R}^{n+1})$.

The answer is yes:

Theorem ("no retraction of B^{n+1} onto S^n"): There is no continuous function $f : B^{n+1} \to S^n$ such that $f(\vec{v}) = \vec{v}$ for all $\vec{v} \in S^n$ $(\subset B^{n+1})$.

The idea of the proof is to replace winding number in the $n=1$ case with a idea that applies to $n>1$ cases but is otherwise similar. In particular, there is a way to assign a number (an integer) to each map $f : S^n \to S^n$ in such a way that (1) homotopy invariance still holds: if $H : S^n \times [0,1] \to S^n$ is continuous, then degree of $H(\cdot\, , s) : S^n \to S^n$ is independent of s;

(2) degree of identity map of S^n to S^n is 1

(3) degree of a constant map $S^n \to$ (one point of) S^n is 0.

The "no retraction" result is then proved as before: If $f : B^{n+1} \to S^n$ were a retraction, then
$H(\vec{v}, s) = f((1-s)\vec{v}) \qquad \vec{v} \in B^{n+1}, s \in [0,1]$
would give a contradiction of (1), (2) and (3) combined. We shall discuss in a moment how to define the degree so that (1), (2) & (3) happen.

(Note: In case $n=1$, degree and winding number mean the same thing. The interest is in how to do the $n>1$ cases).

Meanwhile if we assume the "no retraction" theorem, we get the following interesting result:

Theorem ("Brouwer Fixed Point Theorem"): If $f: B^{n+1} \to B^{n+1}$ is a continuous function, then $\exists\ \vec{v} \in B^{n+1}$ such that $f(\vec{v}) = \vec{v}$.

Slogan: "Every continuous function from B^{n+1} to B^{n+1} $(n \geq 1)$ has a fixed point".

This is actually when $n=1$ also: $B^1 = [-1,1] \subset \mathbb{R}$ and every continuous function from $[-1,1]$ to itself does have a fixed point (exercise).

Proof: Suppose $f: B^{n+1} \to B^{n+1}$ has no fixed point. For each $\vec{v} \in B^{n+1}$, define $R(\vec{v})$ = the unique point of intersection of the line segment (extended) from $f(\vec{v})$ to $\vec{v}$ with S^n. Specifically
$R(\vec{v}) = f(\vec{v}) + \lambda_0(\vec{v} - f(\vec{v}))$
where λ_0 is the unique number ≥ 0 (actually >0) with
$$\| f(\vec{v}) + \lambda_0 (\vec{v} - f(\vec{v})) \| = 1.$$
(The definition here uses that $\vec{v} - f(\vec{v}) \neq \vec{0}$!)

It is elementary to see that R is defined and continuous on B^{n+1} and that $R(\vec{v}) = \vec{v}$ if $\vec{v} \in S^n$. Thus R is a retraction of B^{n+1} onto S^n. This contradiction shows that no such fixed point-free $f: B^{n+1} \to B^{n+1}$ exists. □

Now we outline in general terms how to define degree of $f: S^n \to S^n$. (See Gamelin & Greene, Chapter 4 for details). We assume first that f is differentiable: Defining differentiability

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here can be made easy by extending f to $F: \{\vec{v} : \frac{3}{4} < \|v\| < 1\frac{1}{4}\} \to \mathbb{R}^{n+1}$

by setting $F(\vec{v}) = \|v\| \, f\left(\frac{v}{\|v\|}\right)$.

Then we say f is differentiable if F is. The mapping F (assuming f is differentiable so that F is) has a Jacobian as usual: $\det(\partial F_i / \partial x_j)$ if $F = (F_1, \ldots, F_{n+1})$ in components.

We define $\deg f = \int_{S^n}$ Jacobian of F

where the integral is with respect to the usual "area" integral on S^n (this can also be reduced to an $\mathbb{R}^{n+1}$ item if desired).

This works, though it is not clear that degree f is an integer. It is, though: degree f = the number of times a "typical" point of S^n occurs as an f-image, counting orientation-reversed times negatively. Details of this would carry us too far afield. This subject is treated in books on differential topology, c.f. Guillemin & Pollack as well as Chap 4 of Gamelin & Greene.

To define degree of f for f continuous, we approximate f by a differential map and show that degree of the approximation is independent of which differentiable approximation is used, if the approximation is sufficiently close.

This is only a broad outline. The details require some work! But that is for another time.