Why All Norms on a Finite-Dimensional Vector Space are Equivalent

Definition: Two norms $\| \ \|_1$ and $\| \ \|_2$ on a vector space V are equivalent if $\exists\, C_1, C_2 > 0$ such that for all $v \in V$

$$\|v\|_1 \leq C_1 \|v\|_2 \ \& \ \|v_2\| \leq C_2 \|v\|_1 .$$

Theorem: If V is finite-dimensional, then every two norms $\| \ \|_1$ and $\| \cdot \|_2$ are equivalent.

Proof: It is enough to take $V = \mathbb{R}^n$ and one of the norms to be $\|(x_1, \ldots x_n)\|_1 = |x_1| + |x_2| + \ldots + |x_n|$.
(Reason: Every finite-dimensional vector space is isomorphic to $\mathbb{R}^n$ some n. And if every norm on $\mathbb{R}^n$ is equivalent to $\| \ \|_1$ then every pair of norms are equivalent because two norms both equivalent to a third norm are equivalent to each other [Exercise]).
Let $\| \ \|$ be a norm other than $\| \ \|_1$ on $\mathbb{R}^n$. Then

$$\|(x_1, x_2 \ldots, x_n)\| = \|\Sigma x_i e_i\|$$

$$\leq \sum_{i=1}^{n} |x_i| \|e_i\| \leq \left(\max_{i=1,\ldots,n} \|e_i\|\right) \|(x_1, \ldots, x_n)\|$$

where e_i = the vector with all components $= 0$ except for a 1 in the ith slot.

Now consider the function

$$(\mathbb{R}^n, \| \ \|_1 \text{ metric}) \xrightarrow{I} (\mathbb{R}^n, \| \ \| \text{ metric})$$

that sends $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_n)$:
That is, the function is the identity, but the metric changes (like the function on $C([0,1])$ with sup norm to the ~~fam~~ space $C([0,1])$ with l_2 norm, for example, that we have considered earlier).

This function T is continuous: to see this suppose $v_j \to v_0$ (convergent sequence) in $(\mathbb{R}^n, \| \ \|_1 \text{ norm})$. Then $\|Tv_j - Tv_0\|$
$= \|T(v_j - v_0)\| \leq C \|v_j - v_0\|$ where
$C = \max_{i=1,\ldots n} \|e_i\|$. So $\|Tv_j - Tv_0\| \to 0$ as

$j \to +\infty$. So T is continuous.

In particular, $\{v \in V : \|v\|_1 = 1\}$ is a compact set in $(\mathbb{R}^n, \| \ \|)$ metric because it is image under the continuous map T of the set $\{v \in V : \|v\|_1 = 1\}$ which is compact in $(\mathbb{R}^n, \| \ \|_1)$ by the Heine-Borel Theorem. The function $v \to \|v\|$ is continuous on $(\mathbb{R}^n, \| \ \|)$, by the triangle inequality (or the sublinear property (iii) of $\| \ \|$). And it is positive except if $v = 0$. In particular, it is positive on $\{v \in \mathbb{R}^n : \|v\|_1 = 1\}$ since $\vec{0} \notin$ this set. By compactness, $\exists\ c > 0$ such that $\|v\| \geq c$ for all $v \in \{v \in \mathbb{R}^n : \|v\|_1 = 1\}$.
So $\|v\| \geq c \|v\|_1$ for all $v \in \{v \in \mathbb{R}^n : \|v\|_1 = 1\}$.
Now if $w \neq \vec{0} \in \mathbb{R}^n$, then
$w = \|w\|_1 \frac{w}{\|w\|_1}$ and $\left\| \frac{w}{\|w\|_1} \right\| = 1$. So

$\|w\| \geq c\|w\|_1$ because $\|w\| = \|w\|_1 \left\| \frac{w}{\|w\|_1} \right\| \geq c\|w\|_1 \cdot \left\| \frac{w}{\|w\|_1} \right\|_1 = c\|w\|_1$

So for all $w \in \mathbb{R}^n$

$$\|w\| \geq c\|w\|_1 .$$ (This is obvious if $w = \vec{0}$ so our $w \neq \vec{0}$ case suffices)

Hence

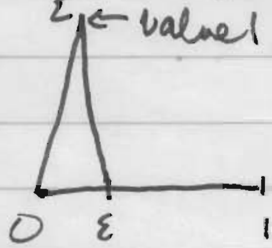
$$\|w\|_1 \leq \left(\frac{1}{c}\right)\|w\| \quad \text{for all } w \in \mathbb{R}^n$$

Since we already had an estimate the other way

$$\|v\| \leq \left(\max_{i=1,\dots n}(\|e_i\|)\right)\|v\|_1 ,$$

the equivalence of $\| \ \|$ and $\| \ \|_1$ is established. □

Note that the sup norm and the L^2 norm on $C([0,1])$ are not equivalent; If f_ε $0<\varepsilon<1$ is the function



then sup norm of $f_\varepsilon = 1$ but L^2 norm of $f_\varepsilon \to 0$ as $\varepsilon \to 0$.

Finite-dimensionality is essential in our theorem (Exercise! $C([0,1])$ is not finite dimensional).