Why Finite Dimensional Subspaces are always closed and Banach spaces cannot be countably infinite dimensional.

Definition: A normed vector space $(V, \| \ \|)$ is <u>complete</u> if V is a complete metric space in the metric defined by $d(v_1, v_2) = \| v_1 - v_2 \|$.

Lemma: If $\| \ \|_1$ and $\| \ \|_2$ are both norms on a vector space V and if $\| \ \|_1$ and $\| \ \|_2$ are equivalent, then $(V, \| \ \|_1)$ is complete if and only if $(V, \| \ \|_2)$ is complete.

Proof: Suppose $\| \ \|_2$ is complete. To prove that $\| \ \|_1$ is complete, let $\{v_i : i = 1, 2, 3 \ldots\}$ be a Cauchy sequence in $(V, \| \ \|_1)$, i.e., for each $\varepsilon > 0$, $\exists\, N_\varepsilon \ni i, j \geq N_\varepsilon \Rightarrow \| v_i - v_j \|_1 < \varepsilon$. Since $\| \ \|_1$ and $\| \ \|_2$ are equivalent, $\exists\, C_2 > 0$ such that $\| v \|_2 \leq C \| v \|_1$ for all $v \in V$. It follows that $\{v_i\}$ is a Cauchy sequence in $(V, \| \ \|_2)$, namely (for $\varepsilon > 0$) $i, j \geq N_{\varepsilon / C_2} \Rightarrow$
$\| v_i - v_j \|_2 \leq C_2 \| v_i - v_j \|_1 < C_2 \cdot \frac{\varepsilon}{C_2} = \varepsilon$.
Thus $\exists\, v_0$ such that $\{v_i\}$ converges to v_0 in $\| \ \|_2$ (by the assumed completeness of $(V, \| \ \|_2)$), or equivalently $\lim_{i \to \infty} \| v_i - v_0 \|_2 = 0$. Since $\| \ \|_1$ & $\| \ \|_2$ are equivalent, $\exists\, C_1$ such that $\| v \|_1 \leq C_1 \| v \|_2$ for all $v \in V$. Hence $\| v_i - v_0 \|_1 \leq C_1 \| v_i - v_0 \|_2$. Thus $\lim_{i \to \infty} \| v_i - v_0 \|_1 = 0$ and $\{v_i\}$ converges to v_0 in $\| \ \|_1$ $\square$

Lemma: If $(V, \| \ \|)$ is a normed vector space (over $\mathbb{R}$) and V is finite-dimensional, then $(V, \| \ \|)$ is complete.

Proof: Suppose $\{e_1, \ldots, e_n\}$ is a basis for V. Then the norm $\| \sum \alpha_i e_i \| = \sum_{i=1}^{n} |\alpha_i|$ is complete by elementary considerations (and the completeness of $\mathbb{R}$). Since all norms are equivalent on a finite-dimensional vector space (over $\mathbb{R}$), $(V, \| \ \|)$ is complete by the previous. $\square$

Theorem: If $(V, \| \ \|)$ is a normed vector space (over $\mathbb{R}$) and if W is a finite-dimensional vector subspace of V, then W is closed in V with respect to the metric on V defined by

$$d(v_1, v_2) = \| v_1 - v_2 \| , \quad v_1, v_2 \in V.$$

Proof: The norm that W inherits from V and its $\| \ \|$ norm is complete by the previous lemma. So $(W, d|_{W \times W})$ is a complete metric space. But a complete subspace of a metric space is always closed: a sequence in the subspace that converges to a point in the whole space is a Cauchy sequence in the whole space and is hence Cauchy in the subspace. Completeness (of the subspace) implies it converges to a point in the subspace. Thus the subspace contains all its own sequential limits and is hence closed in the whole space. $\square$

Lemma: If a (vector) subspace W of a normed vector space $(V, \| \ \|)$ has nonempty interior, then $W = V$.

Proof: Suppose for some $w_0 \in W$, there is an open ball $B(w_0, \varepsilon)$, $\varepsilon > 0$ with $B(w_0, \varepsilon) \subset W$. Then, since W is a vector subspace the set $\{v \in W : v = w - w_0,\ w \in B(w_0, \varepsilon)\} \subset W$. But this set $= B(\vec{0}, \varepsilon)$, the open ball in V around 0 of radius ε. Hence $\alpha B(0, \varepsilon) \subset W$ for every $\alpha \in \mathbb{R}$. But

$$\bigcup_{\alpha \in \mathbb{R}} B(0, \varepsilon) = V. \quad \square$$

Corollary: A closed proper subspace $W \subset V$ is nowhere dense (proper means $W \neq V$).

<u>Theorem</u>: If V is a Banach space (complete normed vector space) then either (i) V is finite dimensional or there is no countable (ii) for every sequence of vectors $v_1, v_2, v_3, \ldots$ in V there is a vector $v_0 \in V$ such that v_0 is not a finite linear combination of (some of) the v_i's. So a Banach space that is infinite dimensional cannot have a countable basis.

Proof: Set W_n = the linear span of $v_1, \ldots, v_n$. Without loss of generality, and without altering the meaning of the conclusion, we can assume that each $v_n \notin W_{n-1}$. The subspaces W_n are all closed because they are finite-dimensional. So either $W_n = V$ for some n and V is finite-dimensional <u>or</u> by the Baire Category Theorem $V \neq \bigcup_1^\infty W_n$. $\square$

Note that the Theorem just proven enables one to find normed vector space which are not only not complete in their given norm but are not complete in <u>any</u> norm: they are not "Banachable" simply on account of their vector space structure.

Consider for example the space of all sequences $\{x_i : x_i \in \mathbb{R}, i = 1, 2, 3 \ldots\}$ which are "eventually 0" i.e. $x_i = 0$ for all $i \geq N$, for some N. Set e_j = the sequence with all elements 0 except the jth element, and the jth element $= 1$, e.g., $e_1 = (1, 0, 0, 0 \ldots)$
$e_2 = (0, 1, 0, 0 \ldots)$, $e_3 = (0, 0, 1, 0, 0 \ldots 0)$.
Then every eventually 0 sequence $\{x_i\}$
$= \sum_{i=1}^{\infty} x_i e_i$ and this sum is actually finite. (Moreover, the e_j's are actually linearly independent: they form a basis for the space of eventually 0 sequences). The existence of this set of e_j's and the property indicated shows that the space of eventually 0 sequences cannot be given a norm such that it is complete in that norm.

The reader may find it interesting to consider why the various, natural norms $\sum |x_i|$, $\max |x_i|$, $(\sum (x_i)^2)^{\frac{1}{2}}$ for $\{x_i\}$-norm are none of them complete.