Urysohn's Lemma for Metric Spaces.

Urysohn's Lemma: If X is a normal topological space and E, F are disjoint closed subsets of X, then $\exists$ $f : X \to [0,1]$ continuous with $f \equiv 0$ on E and $f \equiv 1$ on F.

The text gives a proof of this. But the proof is considerably simpler when X is a metric space. So we prove this case first, before turning to the general situation. For this, suppose (X, d) is a metric space and E, F are closed (nonempty) subsets of X with $E \cap F = \phi$. (The case where one of E or $F = \phi$ is trivial so we suppose $E \neq \phi$, $F \neq \phi$).

Set $d(x,E) = \inf_{y \in E} d(x,y)$ and $d(x,F) = \inf_{y \in F} d(x,y)$.

Note that these infs are $< +\infty$ since E and F are both nonempty. Since E is closed, $d(x,E) > 0$ if $x \notin E$, and similarly for F. Since $E \cap F = \phi$, $d(x,E) + d(x,F) > 0$ for every $x \in X$. Now $d(x,E)$ and $d(x,F)$ are continuous on X. Indeed, the triangle inequality gives $|d(x_1,E) - d(x_2,E)| \leq d(x_1,x_2)$ and similarly for $d(\cdot, F)$. So

$$f(x) = d(x,E) / [d(x,E) + d(x,F)]$$

is continuous on X. And clearly $f \equiv 0$ on E, $\equiv 1$ on F $\square$