Sample Problems for Midterm II:
Topological Spaces and Normed Vector Spaces

1. If $S \subset X$, X a topological space, then (definitions)
interior of S = union of all open sets $U \subset S$
closure of S = intersection of all closed sets A with $S \subset A$.

(a) Prove interior of S is open and closure of S is closed

(b) Prove: interior of $X - S = X$ − closure of S)
closure of $X - S = X$ − interior of S).

2. (a) Define what it means for a topological space to be Hausdorff.

(b) Prove that a compact subset S of a Hausdorff topological space is necessarily closed in X.

3. Let X be an infinite set with the "cofinite" topology [a set in X is open if it is ϕ or $= X$ or has finite, nonempty complement].

(a) Prove that X is not Hausdorff.

(b) Exhibit a compact set in X that is not closed.

4. Let X be a Hausdorff topological space, p a point in X and K a compact subset of X with $p \notin K$. Prove that $\exists$ open sets U, V with $p \in U$, $K \subset V$, and $U \cap V = \phi$.

5. Let X be a Hausdorff space and K_1, K_2 compact subsets of X with $K_1 \cap K_2 = \phi$. Prove that $\exists$ open sets U, V with $K_1 \subset U$, $K_2 \subset V$ and $U \cap V = \phi$.

6. Prove that a closed subset of a compact topological space is compact. [Suggestion: If $K \subset X$, K closed, X compact and $K \subset \bigcup_{\lambda \in \Lambda} U_\lambda$, then $X - K$, $U_\lambda, \lambda \in \Lambda$ is an open cover of X].

7. If X is a compact Hausdorff space, then X is normal. [Suggestion: Combine problems 5 & 6].

8. Suppose X is a T_1 topological space [T_1 means one-point sets are necessarily closed]. And suppose for every pair of closed sets E, F with $E \cap F = \phi$, $\exists$ a continuous function $f: X \to [0,1]$ such that $f(x) = 0 \ \forall x \in E$, $f(x) = 1$ for all $x \in F$. Prove that X is normal. [This is the converse of Urysohn's Lemma].

9. Formulate and prove a similar converse for the Tietze Extension Theorem.

10. Prove that for any topological space X, the uniform limit of a sequence of functions $f_i : X \to \mathbb{R}$ is continuous if each f_i is continuous [i.e., if f_i are all continuous and $\{f_i\}$ converges uniformly to $f_0 : X \to \mathbb{R}$ then f_0 is continuous].

11. Let V be a finite-dimensional vector space with norms $\| \ \|_1$ and $\| \ \|_2$.
(a) Define what it means for $\| \ \|_1$ and $\| \ \|_2$ to be equivalent.
(b) Prove that $\| \ \|_1$ and $\| \ \|_2$ are equivalent, (for every pair of norms $\| \ \|_1$ and $\| \ \|_2$)

12. Give an example of a vector space V with two norms $\| \ \|_1$ and $\| \ \|_2$ that are not equivalent and prove the nonequivalence (according to prob 11, V must be infinite dimen.)

13. Prove that the unit ball in a finite-dimensional normed vector space is necessarily compact.

14. Prove that if V is a normed vector space and W is a finite-dimensional subspace of V, then W is a closed subset of V (in the metric space topology on V determined by $d(v_1, v_2) = \| v_1 - v_2 \|$).

15. Give an example of a normed vector space V and a subspace W such that W is not a closed subset of V.

16. Take as given: Every vector space has a basis, that is, a set $\{v_\lambda : \lambda \in \Lambda\}$ such that the set is linearly independent and every $v \in$ the vector space is a finite linear combination of some of the v_λ. Prove: Every vector space has a norm.

17. Suppose V is a vector space that is infinite dimensional but has a countable basis $v_1, v_2, v_3 \ldots$

Show that there is no norm $\| \ \|$ on V such that V is complete in $\| \ \|$ (i.e. Cauchy complete relative to the metric $d(w,u) = \|w-u\|$, $w, u \in V$) [Suggestion: Use Baire Category Theorem]

18. Give a specific example of an infinite-dimensional vector space with a countable basis.

19 Show that if W is a finite-dimensional subspace of a normed vector space V, $\| \ \|$, and if $x \in V$, $x \notin W$, then $\exists\, w_0 \in W$ such that $\|x - w_0\| = \inf_{w \in W} \|x - w\|$

[Suggestion: Only $w \in W$ with $\|w\| \leq 2\|x\|$ need be considered since $\|w\| > 2\|x\| \Rightarrow \|x - w\| > \|x - 0\|$. Use compactness of closed balls in finite dimensional space from here].

20. Use problem 19 to show that if $\{v \in V : \|v\| \leq 1\}$ is compact, then V is Finite dimensional

V normed vector space as usual.