How Urysohn's Lemma implies the Tietze Extension Theorem: the Pictoral Version

Note: The proof as such is given in detail in the Gamelin & Greene textbook. The purpose of the following is to offer a mnemonic picture of how the proof works, not to rewrite the proof itself as such. Read the book! for the detailed proof.

We <u>assume</u> here Urysohn's Lemma: If E, F are closed subsets of a normal topological space X with $E \cap F = \phi$, then $\exists$ a continuous function $f: X \to \mathbb{R}$ such that $f \equiv 0$ on E, $f \equiv 1$ on F and $0 \leq f \leq 1$ everywhere on X. [Note: We have proved this for metric spaces: $f(x) = d(x,E) / (d(x,E) + d(x,F))$ works. The proof for normal topological spaces will be done later. See also the G&G text].

The pictorial illustration of the proof of the following theorem will be done for $X = \mathbb{R}$. The theorem itself is quite general:

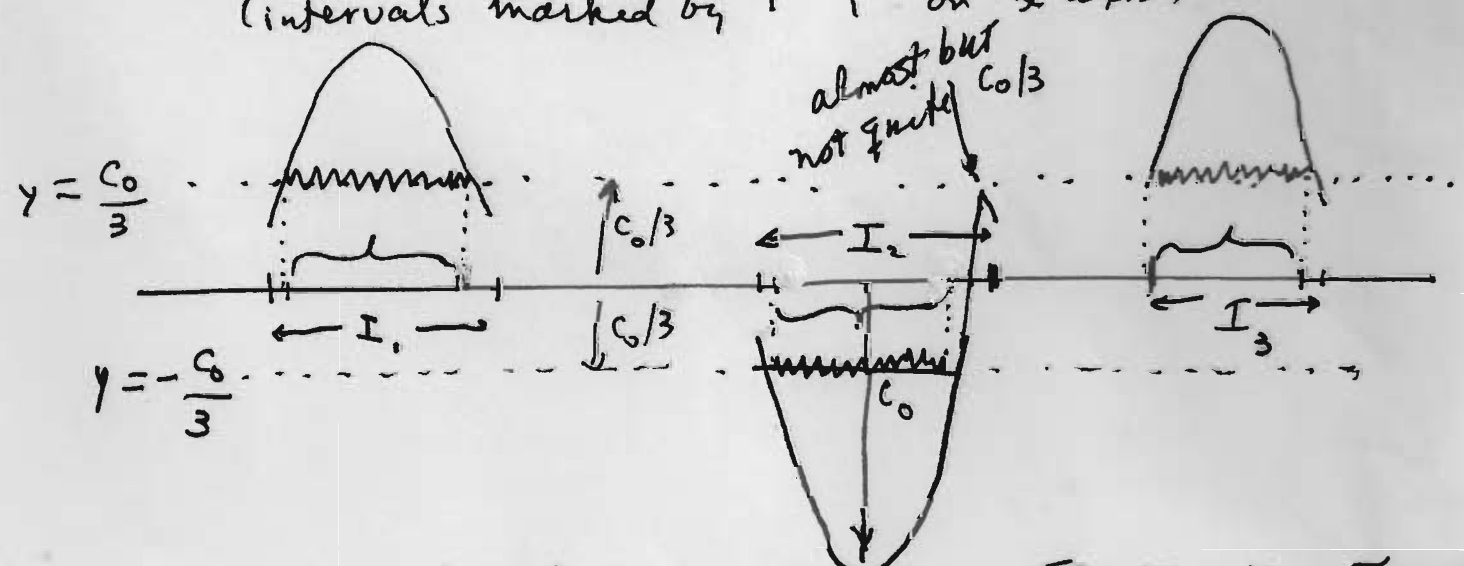
Tietze Extension Theorem: If X is a normal topological space and E is a (nonempty) closed subset of X and $f: E \to \mathbb{R}$ is a continuous function, then there is a continuous function $\hat{f}: X \to \mathbb{R}$ such that $\hat{f}(x) = f(x)$ for each $x \in E$.

The general case of this theorem follows from the theorem for bounded f: see Exercise 6, p.78 in Gamelin & Greene. We shall illustrate pictorially the proof for the case f bounded. i.e. $\exists\ c_0 > 0 \ni |f(x)| \le c_0$, $\forall x \in E$.

[Note: For E closed in $\mathbb{R} = X$, the theorem can be easily proved by linear interpolation on the open intervals that are the connected components of $\mathbb{R} \setminus E$. (Homework VI, Exercise 1). The following proof is presented because it illustrates the general proof and it is possible to present it pictorially].

The proof for the bounded case: $|f(x)| \le c_0$, $\forall x \in E$. illustrated for a particular function and a particular closed set E:

Graph of f defined on $I_1 \cup I_2 \cup I_3$ (intervals marked by ⊢ ⊣ on x axis)



$c_0 = \sup_{x \in E} |f(x)|$

$E = I_1 \cup I_2 \cup I_3$

Consider the sets $\{x \in E : f(x) \geq \frac{c_0}{3}\}$.
In the picture, this set consists of the two intervals on the x-axis marked by . The set $\{x \in E : f(x) \leq -\frac{c_0}{3}\}$ is the set (interval) marked by .

Now in general not just in our pictorial case $\{x \in E : f(x) \geq \frac{c_0}{3}\}$ is closed in E (by the continuity of f on E); and since E is closed in X by hypothesis, the set $\{x \in E : f(x) \geq \frac{c_0}{3}\}$ is also closed in X.
Similarly $\{x \in E : f(x) \leq -\frac{c_0}{3}\}$ is closed in X.

By Urysohn's Lemma, we can find a function h_1, continuous on X, such that
$h_1 = 1$ on $\{x \in E : f(x) \geq \frac{c_0}{3}\}$ and
$h_1 = 0$ on $\{x \in E : f(x) \leq -\frac{c_0}{3}\}$ and $0 \leq h_1 \leq 1$ everywhere.

Set $g_1 = \frac{2c_0}{3}\left(h_1 - \frac{1}{2}\right)$.

Then $-\frac{c_0}{3} \leq g_1 \leq \frac{c_0}{3}$ everywhere

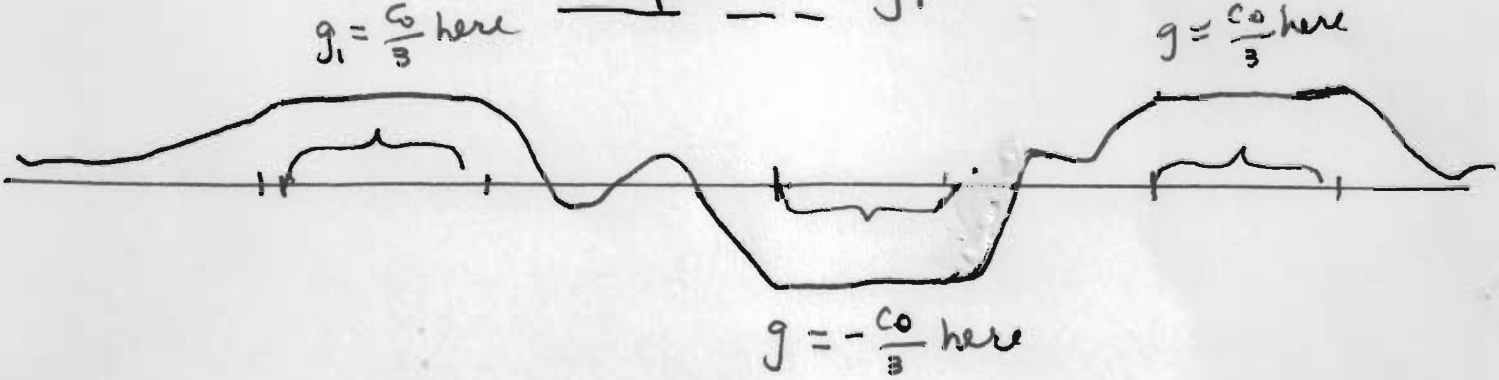
and $g_1 = \frac{c_0}{3}$ on $\{x \in E : f(x) \geq \frac{c_0}{3}\}$

while $g_1 = -\frac{c_0}{3}$ on $\{x \in E : f(x) \leq -\frac{c_0}{3}\}$.

Case by case analysis shows
$$|f(x) - g_1(x)| \leq \frac{2}{3} c_0 \quad \text{for } \forall\, x \in E.$$

This illustrates the situation: one possible g_1 of many is shown

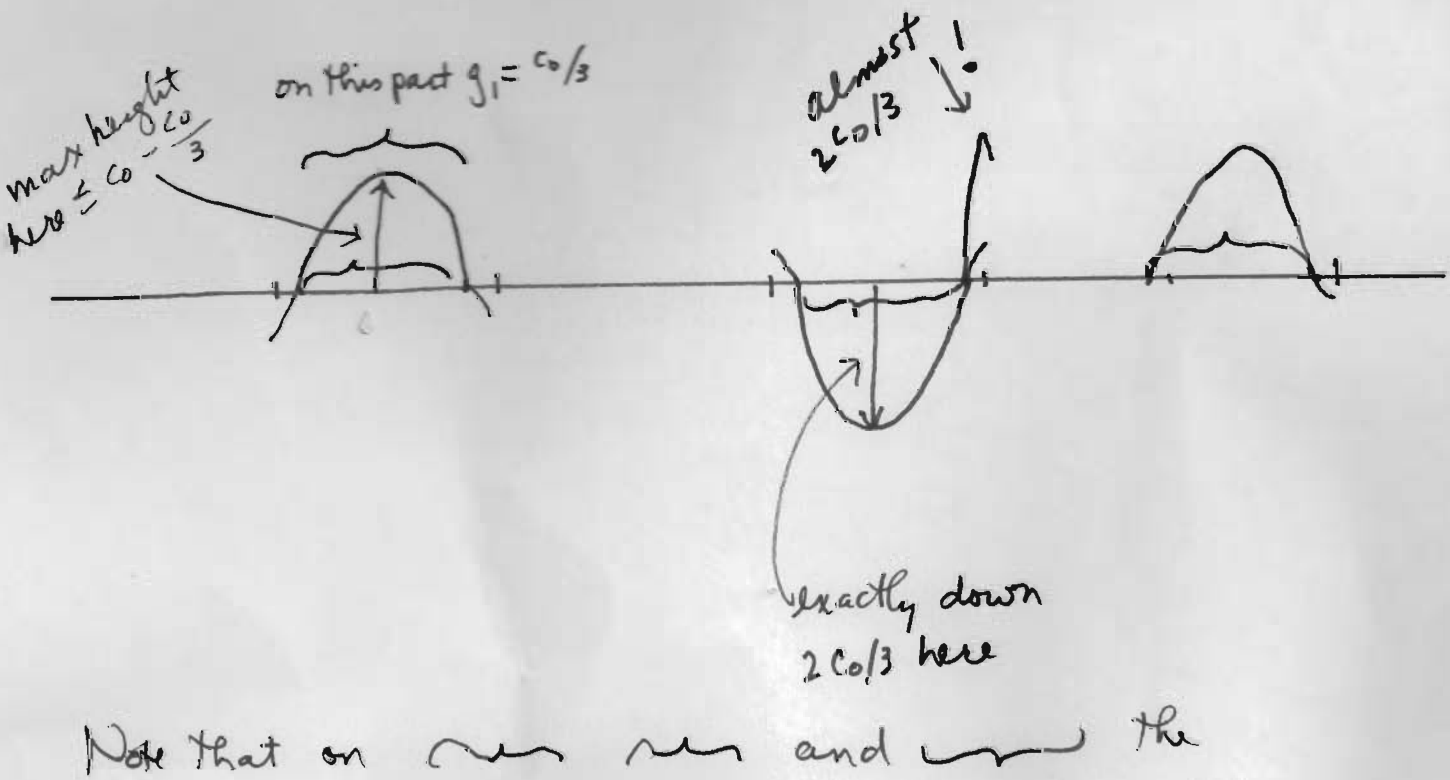
Graph of a g_1



Note that off of $\frown \cup \frown \cup \smile$,

g_1 is unspecified except that $|g_1| \leq c_0/3$.

Here is $f(x) - g_1(x)$, $x \in E$ in the cases of the point $x \in I_1$ or I_2 or I_3 ($E = I_1 \cup I_2 \cup I_3$)



Note that on $\frown$ $\frown$ and $\smile$ the f graph is moved by exactly $c_0/3$ (down in first two cases, up in last) in forming $f - g_1$. But off the $\frown$ intervals, we are not sure what happens except that we are not further from the x-axis with it

than $2c_0/3$ anywhere on E.

Summary: $|f - g_1| \leq \frac{2}{3} c_0$ on E

$|g_1| \leq \frac{1}{3} c_0$ everywhere.

Now we do the process again with f (on E) replaced by $f - g_1$ to get g_2 such that

$$|(f - g_1) - g_2| \leq \frac{2}{3}\left(\frac{2}{3} c_0\right) \text{ on } E$$

$$|g_2| \leq \frac{1}{3}\left(\frac{2}{3} c_0\right) \text{ everywhere.}$$

Inductively: $\exists\ g_1, g_2, g_3 \ldots g_n \ni$

$$|f - \sum_{i=1}^{n} g_i| \leq \left(\frac{2}{3}\right)^n c_0 \text{ on } E$$

$$|g_n| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} c_0$$

Then $\sum_{i=1}^{+\infty} g_i$ converges uniformly on X to a continuous function $\hat{f} : X \to \mathbb{R}$ (because $\sum_{n=1}^{+\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} < +\infty$: Homework exercise 1, Hwk. VI).