Solutions to Homework VI

1. Given $\varepsilon > 0$, choose N such that
$|f_N(x) - f_0(x)| < \varepsilon/3$ for all $x \in X$ (possible by uniform convergence). Since f_N is continuous, given $x_0 \in X$, $\exists\ U_0$ open in X such that $x_0 \in U_0$ and $x \in U_0 \Rightarrow |f_N(x) - f_N(x_0)| < \varepsilon/3$.
With N and U_0 so chosen, if $x \in U_0$ then

$$|f_0(x) - f_0(x_0)| \le |f_0(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_0(x_0)|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

So $x \in U_0 \Rightarrow |f_0(x) - f_0(x_0)| < \varepsilon$, and f_0 is continuous at x_0, which was an arbitrary point of X □

2. Given: $f \overset{\text{cont.}}{:} E \to \mathbb{R}$, E closed in a normal topological space X. To find: $\hat{f} \overset{\text{cont.}}{:} X \to \mathbb{R}$ such that $\hat{f}(x) = f(x)$, $\forall x \in E$.

Consider the composition $H \circ f : E \to [-1, 1]$, where $H(\alpha) = \frac{2}{\pi} \arctan \alpha$, $\alpha \in \mathbb{R}$ (so that $H : \mathbb{R} \to (-1, +1)$). By the bounded function Tietze Extension Theorem, $\exists\ F : X \to \mathbb{R}$ such that $F(x) = (H \circ f)(x)$ if $x \in E$. By composing F (if necessary) with the function
$x \to x$ if $x \in [-1, 1]$, $x \to -1$ if $x < -1$,
$x \to +1$ if $x > +1$, we can and do assume $F(X) \subset [-1, +1]$ and still $F(x) = H(f(x))$ if $x \in E$.
Let C = the set of $x \in X$ such that $F(x) = +1$ or $F(x) = -1$. Note that C is closed and disjoint from E since, if $x \in E$, $H(f(x)) \in (-1, 1)$ while $F(x) = H(f(x))$. Choose (by Urysohn's Lem.)

a continuous function $\lambda : X \to [0,1]$ such that $\lambda(x) = 0$ if $x \in C$ but $\lambda(x) = 1$ if $x \in E$.

Then $\lambda * F : X \to (-1, +1)$ because $\lambda = 0$ when $|F(x)| = 1$ so that

$$|(\lambda * F)(x)| = |\lambda(x)||F(x)| < 1 \quad \text{for all } x \in X.$$

Consider $\tan\left(\frac{\pi}{2}\lambda(x)\, F(x)\right)$ as a function of $x \in X$.

It is continuous since tan is continuous on $(-1, +1)$ and $\lambda(x) F(x) \in (-1, +1)$ for all $x \in X$.

If $x \in E$, then $\lambda(x) = 1$ & $F(x) = \frac{2}{\pi}\arctan f(x)$

so

$$\tan\left(\frac{\pi}{2}\lambda(x) F(x)\right) = \tan\left(\frac{\pi}{2}\cdot\frac{2}{\pi}\arctan f(x)\right) = f(x).$$

So $x \to \tan\left(\frac{\pi}{2}\lambda(x) F(x)\right)$ does the job for $\hat{f}$ $\square$.

3. Let $F : E \to \mathbb{R}^n$, E closed in a normal topological space, F continuous. Then we can write $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, $x \in E$, where $f_i : E \to \mathbb{R}$, $i = 1, 2, \ldots, n$ are (uniquely determined by F) continuous functions. By problem 2, there are continuous functions $\hat{f}_i : X \to \mathbb{R}$, $i = 1, 2, \ldots, n$ such that $\hat{f}_i(x) = f_i(x)$ (for each i and) $x \in E$. Define $\hat{F} : X \to \mathbb{R}^n$ by

$$\hat{F}(x) = (\hat{f}_1(x), \hat{f}_2(x), \ldots, \hat{f}_n(x)).$$

Then $\hat{F}$ is continuous on X, and $\hat{F}(x) = F(x)$ if $x \in E$.

$\square$

4. Let $\gamma : [0,1] \to Y$ be a path from p to q, i.e. γ is continuous and $\gamma(0) = p$ and $\gamma(1) = q$. Choose by Urysohn's Lemma ~~and~~ a continuous function $f_0 : X \to [0,1]$ such that $f_0(x) = 0$ if $x \in E$ while $f_0(x) = 1$ if $x \in F$. Then set $f_{pq}(x) = \gamma(f_0(x))$ for all $x \in X$.

Clearly f_{pq} is continuous since γ is and so is f_0 and range $f_0 \subset$ domain of γ. If $x \in E$ so that $f_0(x) = 0$, then $f_{pq}(x) = \gamma(0) = p$. Similarly if $x \in F$, then $f_{pq}(x) = \gamma(1) = q$. $\square$

5. Since S^1 is compact (Heine-Borel Theorem), $h(S^1)$ is compact in X, being the image in X of the compact set S^1 under the continuous function h. ~~[illegible]~~ Since X is normal, it is Hausdorff. So every compact subset of X is closed. In particular, $h(S^1)$ is closed. Define $F : h(S^1) \to \mathbb{R}^2$ by $F(p)$ $=$ the (x,y) coordinate point in $\mathbb{R}^2$ $= h^{-1}(p)$: h^{-1} formally sends $h(S^1)$ onto S^1, but since $S^1 \subset \mathbb{R}^2$, we can consider it as function into $\mathbb{R}^2$ instead of onto S^1. We call this F. By problem 3, $\exists\ \hat{F} : X \to \mathbb{R}^2$ such that, for each $p \in h(S^1)$, $\hat{F}(p) = F(p) = h^{-1}(p)$. Now let $U_0 = \mathbb{R}^2 - \{(0,0)\}$ and define $R : U_0 \to S^1$ by $R(\vec{v}) = \frac{\vec{v}}{\|\vec{v}\|}$. Then consider the function $f : F^{-1}(U_0) \to S^1$

defined by $f(q) = h(R(F(q)))$, $q \in F^{-1}(U)$.
Note that $F^{-1}(U)$ is continuous since F is continuous on X and U is open (in $\mathbb{R}^2 \supset$ range F). Also f is defined because
$$F(F^{-1}(U)) = U \subset \text{domain of } R$$
and range $R = S^1 \subset$ domain of h.
Also if $p \in h(S^1)$, $p = h(\lambda)$, $\lambda \in S^1$,
then $F(p) = \lambda$, $R(F(p)) = R(\lambda) = \lambda$
and $f(p) = h(R(F(p))) = h(\lambda) = p$. So f works.
with $U = F^{-1}(U_0)$, $f: U \to h(S^1)$ as defined. $\square$

6. This works exactly like problem 5:
If $h: S^n \to X$ is a homeomorphism of S^n onto its image in a normal topological space X, then there is a neighborhood U of $h(S^n)$ and a continuous function $f: U \to h(S^n)$ such that $f(x) = x$ if $x \in h(S^n)$.
Proof as before: S^n compact $\Rightarrow$ $h(S^n)$ compact $\Rightarrow$ $h(S^n)$ closed. Tietze Extension as in prob. 3 gives $F: X \to \mathbb{R}^{n+1}$ extending $h^{-1}: h(S^n) \to S^n \subset \mathbb{R}^{n+1}$.
With $U_0 = \mathbb{R}^{n+1} - \{\vec{0}\}$, $U = F^{-1}(U_0)$ is desired U, $f(\cdot) = h(R(F(\cdot)))$ where $R(\vec{v}) = \vec{v}/\|\vec{v}\|$, $\vec{v} \in \mathbb{R}^{n+1}$, $\vec{v} \neq \vec{0}$ as in problem 5 does the job. $\square$

7. Same pattern as number 6. Write $R: U_0 \to Y$ for the problem's $f: U \to Y$ for consistency with problems 5 & 6 notation. If $h: Y \to X$, X normal is continuous, then $h(Y)$ is compact

in X (since Y is compact and h is continuous) so $h(Y)$ is closed in X. By Tietze Extension, $h^{-1}: h(Y) \to Y \subset \mathbb{R}^n$ extends to $F: X \to \mathbb{R}^n$. Then $F^{-1}(U_0)$ is open in X and $h(R(F(\cdot))$ is the desired function on U_0 (which is U in the original notation of the problem).

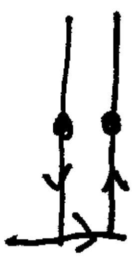
8. Since $[0,1]$ is compact, $h([0,1])$ is compact. So $h([0,1])$ is closed. And $h^{-1}: h([0,1]) \to \mathbb{R}$ has an extension $f: X \to \mathbb{R}$, which we can assume ~~[illegible]~~ has $f(X) \subset [0,1]$ (by composing with $x \to x$, $x \in [0,1]$, $x \to -1$ if $x < -1$, $x \to 1$ if $x > 1$, as before). Set $F(\cdot) = h(f(\cdot))$. $\square$

9. If $f: U \to Y$ is a retraction ($f(x) = x$ for all $x \in Y$, U open, $Y \subset U$), choose $V \subset U$ with $\bar{V}$ compact, $Y \subset V$. This choice is possible since we can first take U bounded and then, since $\mathbb{R}^2$ is normal, take V ~~as~~ with ~~[illegible]~~ $Y \subset V$ and $\bar{V} \subset U$ so $\bar{V}$ is closed and bounded, hence compact. Then $f|V$ is uniformly continuous because $f|\bar{V}$ is uniformly continuous by the compactness of $\bar{V}$.
With V so chosen:
Since $(0, \frac{1}{2}) \in Y$, and $Y \subset V$, there $\exists\ \varepsilon > 0$

such that $B((0,\frac{1}{2}),\varepsilon) \subset V$. By uniform continuity — or even just continuity at $(0,\frac{1}{2})$ — we can also choose $\varepsilon > 0$ so small that the y-coordinate of $f(q)$ is $> \frac{1}{4}$ for all $q \in B((0,\frac{1}{2}),\varepsilon)$ — this is because the y-coordinate of $f((0\frac{1}{2}))$ is $\frac{1}{2}$ since $f((0,\frac{1}{2})) = (0,\frac{1}{2})$ because $(0,\frac{1}{2}) \in Y$.

Now choose $n > 10/\varepsilon$. Then $(\frac{1}{n+1},\frac{1}{2})$ and $(\frac{1}{n},\frac{1}{2})$ both lie in $B((0,\frac{1}{2}),\varepsilon)$ and so the closed line segment between $(\frac{1}{n+1},\frac{1}{2})$ and $(\frac{1}{n},\frac{1}{2})$ also lies in $B((0,\frac{1}{2}),\varepsilon)$.

Now we ask: what is the f-image of this line segment? Clearly, it is a continuous path in Y from $(\frac{1}{n+1},\frac{1}{2})$ to $(\frac{1}{n},\frac{1}{2})$ since f fixes $(\frac{1}{n+1},\frac{1}{2})$ and $(\frac{1}{n},\frac{1}{2})$.
But the only such continuous paths have to contain points with y-coordinate $= 0$,



as shown. If we disallow the possibility that $y=0$ somewhere, then the possible x-coordinate values belong to the set $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots\}$ and hence x is constant along the path!
But $y > \frac{1}{4}$ along the f-image of the line segment from $(\frac{1}{n+1},\frac{1}{2})$ to $(\frac{1}{n},\frac{1}{2})$ by the choice of ε. This is a contradiction. $\square$

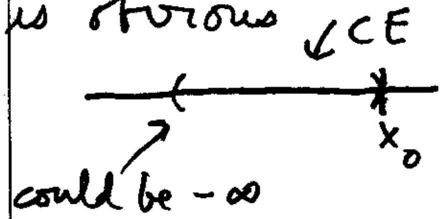
10. If $x \in \mathbb{R}-E$, then x lies in the open interval (α,β) where $\alpha = \sup\{\lambda : \lambda < x, \lambda \in E\}$ and $\beta = \inf\{\lambda : \lambda > x, \lambda \in E\}$. ($\beta$ may be $+\infty$, α may be $-\infty$). Intervals of this form are either coincident or disjoint (obvious) so $\mathbb{R}-E = \underset{\text{finite or countable}}{\bigcup}$ open intervals, disjoint union!

If (α,β) is such a maximal open interval in $\mathbb{R}\setminus E$, we set (when $\alpha,\beta \in \mathbb{R}$)

$$\hat{f}(x) = \frac{x-\alpha}{\beta-\alpha} f(\alpha) + \frac{\beta - x}{\beta-\alpha} f(\beta)$$

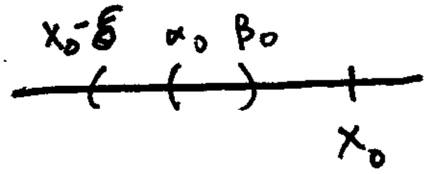
for $f : E \to \mathbb{R}$. (On $(-\infty,\beta)$ if it occurs, we set $\hat{f}(x) = f(\beta)$, $x \in (-\infty,\beta)$ and on $(\alpha,+\infty)$, $\hat{f}(x) = f(\alpha)$, $x \in (\alpha,+\infty)$); set $\hat{f}(x) = f(x)$ if $x \in E$. This defines $\hat{f} : \mathbb{R} \to \mathbb{R}$. We want to show that $\hat{f}$ is continuous. Continuity at a point x_0 in one of the open intervals is clear: $\hat{f}$ is (constant or) linear in an open set around x_0. It remains to check continuity of $\hat{f}$ at x_0, $x_0 \in E$. For this, it suffices to check continuity from each side, that is continuity at x_0 of $\hat{f}|_{(-\infty,x_0]}$ and of $\hat{f}|_{[x_0,+\infty)}$. We do $\hat{f}|_{(-\infty,x_0]}$. The other case is similar.

If x_0 is the right-hand endpoint of one of the maximal intervals in $\mathbb{R}-E$, continuity is obvious since to the left of x_0, $\hat{f}$ is (constant or) linear on some interval with x_0 as right-hand endpoint and by definition $\hat{f}(x)$ on this interval converges to $f(x_0)$ as $x \to x_0^-$.



If $E \supset$ a closed interval of nonzero length with righthand endpoint $= x_0$, continuity is similarly obvious since f is continuous on E and $\hat{f} = f$ on E.

It remains to consider the case where, for each $\delta > 0$, $(x_0 - \delta, x_0)$ contains points of E and points not in E. In this case, given $\varepsilon > 0$, we choose δ such that $|x - x_0| < \delta$ and $x \in E$ (recall x_0 is in E), $|f(x) - f(x_0)| < \varepsilon$. Now there is an (α, β) interval (of the maximal kind) $\subset \mathbb{R}-E$ with $(\alpha, \beta) \subset (x_0 - \delta, x_0)$.



(Otherwise we would not have points of $\mathbb{R}-E$ and points of E arbitrarily close to x_0 but $< x_0$). Claim: With (α_0, β_0) so chosen, $\alpha_0 > x - \delta$
$|\hat{f}(x) - \hat{f}(x_0)| < \varepsilon$ if $x \in (\alpha_0, x_0)$.

Reason: If $x \in E$, this is true by choice of δ.
If $x \notin E$, then $x \in (\alpha_1, \beta_1)$ with

$x_0 - \delta < \alpha_1 < \beta_1 < x_0$. Since $\hat{f}(x)$ is a linear interpolation of $f(\alpha_1)$ and $f(\beta_1)$, both of which are within ε of $f(x_0)$, it follows that $\hat{f}(x)$ is within ε of $f(x_0)$.

This completes the proof of continuity. $\square$