Homework VI : Due Friday, May 29, 2009

1. Prove: If $f_i : X \to \mathbb{R}$ is a sequence of continuous functions $(i = 1, 2, 3 \ldots)$ such that $\{f_i\}$ converges uniformly to a function $f_0 : X \to \mathbb{R}$, then f_0 is continuous. Here:

   (a) X is an arbitrary topological space, no special "separation axioms" assumed

   4 (b) f_i converges uniformly to f_0 means (as usual) given $\varepsilon > 0$, $\exists\, N_\varepsilon \ni i \geq N_\varepsilon \Rightarrow$

   $$|f_i(x) - f_0(x)| < \varepsilon \quad \text{for all } x \in X.$$

   Suggestion: Recall the proof for the metric space (X a metric space) case, the "three term trick".

2. Show how the Tietze Extension Theorem for bounded functions (as we proved it) implies the extension of arbitrary $\mathbb{R}$-valued continuous functions. (cf. Problem 6, p. 78, Gamelin & Greene)

3. State and prove a Tietze Extension Theorem for functions with values in $\mathbb{R}^n$, not just $\mathbb{R}$.

4. Suppose X is a normal topological space and E, F are closed subsets of X with $E \cap F = \phi$. Suppose Y is an arc-wise (pathwise) connected topological space and $p, q \in Y$.

(continued)

4 (continued). Prove that there is a continuous function $f_{pq}: X \to Y$ such that $f_{pq}(x) = p$ if $x \in E$ and $f_{pq}(x) = q$ if $x \in F$.

circle

5. Suppose $h: S^1 \to X$ is a homeomorphism of S^1 $(= \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\})$ onto its image $h(S^1)$ in a normal topological space X.
Prove that there is an open subset U in X with $h(S^1) \subset U$ such that there is a continuous function $f: U \to h(S^1)$ with $f(x) = x$ if $x \in h(S^1)$. [We call f a "retraction" of U onto $h(S^1)$ and say that $h(S^1)$ is a "retract" of U].

Note: h being a homeomorphism onto its image means that h is continuous and one-to-one and that $h^{-1}: h(S^1) \to S^1$ is continuous.

Suggestion: Prove first that $\exists$ a continuous function $F: X \to \mathbb{R}^2$ such that $F(x) = h^{-1}(x)$ if $x \in h(S^1)$. Then look at $F^{-1}(\mathbb{R}^2 - \{(0,0)\})$, which is open in X. If $y \in F^{-1}(\mathbb{R}^2 - \{(0,0)\})$ then $y \to F(y)/\|F(y)\|$ maps y to a point of S^1.

nth dim sphere

Like

6. State and prove a result like that of problem 5 for $S^n = \{(x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1} : \sum_1^n x_i^2 = 1\}$, $n > 1$.

7. A compact set Y in $\mathbb{R}^n$ is called a "neighborhood retract" if there is an open set U in $\mathbb{R}^n$ with $Y \subset U$ and a continuous function $f: U \to Y$ such that $f(y) = y$ if $y \in Y$.
Show that problem 5 can be generalized to all such compact "neighborhood retracts" Y (Y replacing S^1).

8. Show that if $h: [0,1] \to X$, X a normal topological space, is a homeomorphism onto its image, then $\exists$ a continuous function $F: X \to h([0,1])$ such that $h(x) = x$ if $x \in h([0,1])$.

(Like problem 5 except U can be all of X in this case.)

*9. Show that the compact set in $\mathbb{R}^2$

$$Y = \{(0,\lambda) : \lambda \in [0,1]\} \cup \left(\bigcup_{n=1}^{+\infty} \{(\tfrac{1}{n}, \lambda) : \lambda \in [0,1]\}\right)$$
$$\cup \{(\lambda, 0) : \lambda \in [0,1]\}$$

is <u>not</u> a neighborhood retract in the sense of problem 7.

[Suggestion: If Y were a neighborhood retract, $f : U \to Y$ a retraction: choose V^{open} with $Y \subset V^{\text{open}} \subset \bar{V} \subset U$, $\bar{V}$ compact, so f restricted to V is uniformly continuous. Think for n large about what would happen to f along the line segment from $(\frac{1}{n+1}, \frac{1}{2})$ to $(\frac{1}{n}, \frac{1}{2})$. [which will lie in V, for n large].]

10. Suppose E is a closed subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ is a continuous function. Prove directly that $\exists\ \hat{f} : \mathbb{R} \to \mathbb{R}$ continuous with $\hat{f}(x) = f(x)$, $\forall x \in E$ by linear interpolation, using that $\mathbb{R} \setminus E$ = union of maximal open intervals of the form (a,b) or $(-\infty, a)$ or $(b, +\infty)$ and using the interpolation processes shown