Solutions: Homework V

1. Let $\{x_i\}$ be a sequence with $\|x_i\| \leq 1$.
We need to show that $\exists$ a subsequence $\{x_{i_j}\}$ that converges to a point x_0 with $\|x_0\| \leq 1$ — converges in the given norm!
Since the vector space is finite dimensional, it has a finite basis, say $e_1, \ldots, e_n$. Set $\|\sum \alpha_i e_i\|_0 = (\sum_{i=1}^{n} \alpha_i^2)^{\frac{1}{2}}$. This is clearly a norm on the vector space, making it isometric to $\mathbb{R}^n$, standard norm.
We have already shown that $\| \ \|$ and $\| \ \|_0$ are equivalent (every pair of norms on a finite dimensional vector space are equivalent).
So $\|x_i\| \leq 1$, all i, implies $\|x_i\|_0 \leq C$ for some constant C. So there is, (by the Heine-Borel Theorem), a subsequence $\{x_{i_j} : j = 1, 2 \ldots\}$ that converges to some x_0 in the $\| \ \|_0$ norm.
Again by norm equivalence, $\|x_{i_j} - x_0\| \leq C \|x_{i_j} - x_0\|_0$ for some constant C (independent of j), all j. Since $\lim \|x_{i_j} - x_0\|_0 = 0$ it follows that $\lim \|x_{i_j} - x_0\| = 0$ so $\{x_{i_j}\}$ converges to x_0 in $\| \ \|$ norm.

Finally $\big| \|x_{i_j}\| - \|x_0\| \big| \leq \|x_{i_j} - x_0\|$ by sublinearity of the norm (or $\Delta \leq$).
So $\|x_{i_j}\| \leq 1$, all j, $\Rightarrow$ $\|x_0\| \leq 1$,

since $\|x_{i_j} - x_0\|$ goes to 0 as $j \to +\infty$.

We assume, wolog, $x \notin W$ so $x \neq \vec{0}$ in particular.

2. If $w \in W$ and $\|w\| > 2\|x\|$, then $\|w - x\| > \|x\|$ by $\Delta \leq$.

So $\inf_{w \in W} \|w - x\| = \inf_{w \in W,\ \|w\| \leq 2\|x\|} \|w - x\|$.

Now $\{w \in W : \|w\| \leq 2\|x\|\}$ is compact by problem 1. And $w \to \|w - x\|$ is a continuous function on $\{w \in W : \|w\| \leq 2\|x\|\}$ since $\big| \|w_1 - x\| - \|w_2 - x\| \big| \leq \|w_1 - w_2\|$ by $\Delta \leq$. So this continuous function $w \to \|w - x\|$ attains its minimum on $\{w \in W : \|w\| \leq 2\|x\|$. The point at which the minimum is attained is the point required in the problem.

3. Let $V = \mathbb{R}^2$ with $\|(x, y)\| = |x| + |y|$. Then for (x, y) such that $x + y = 1$ & $x \geq 0$ & $y \geq 0$, $d((x, y), (0, 0)) = 1$. It follows that if $W = \{(x, y) : x + y = 0\}$ and $x = (0, -1)$, then

(i) $d((x, y), (0, -1))$, $(x, y) \in W$
$= |x| + |y + 1| = |x| + |1 - x| \geq 1$

(ii) $d((x, y), (0, -1))$ $(x, y) \in W$, $1 \geq x \geq 0$
$= x + |(1 - x)| = x + 1 - x = 1$ (for $x \in [0, 1]$)

So every point $(x, -x)$, $1 \geq x \geq 0$, is a "closest point" in W to $(0, -1)$. Closest points are not unique!

4. If $\|x - w_0\| = \inf_{w \in W} \|x - w\|$

so w_0 is a "closest point" in W to x, then 0 is closest point in W to $x - w_0$ because

$$\inf_{w \in W} \|(x - w_0) - w\| = \inf_{w \in W} \|x - w\|$$

since W is a subspace and $\inf_{w \in W} \|(x - w)\| = \|x - w_0\|$

by choice of w_0. Now for any $\lambda > 0$, $\lambda W \overset{\text{def}}{=} \{\lambda w : w \in W\}$ is W. So if $\vec{0}$ is a closest point in W to $v \in V$, it follows that $\vec{0}$ is a closest point to λv because

$$\|\lambda v\| = \lambda \|v\| = \lambda \inf_{w \in W} \|v - w\|$$

$$= \inf_{w \in W} \|\lambda(v - w)\| = \inf_{w \in W} \|\lambda v - \lambda w\| = \inf_{w \in W} \|\lambda v - w\|$$

(because $\lambda W = W$). Apply this to $v = x - w_0$ to complete the proof, with $\lambda = 1/\|x - w_0\|$.

5. Set $w_1 = v_1 / \|v_1\|$. Inductively: With $w_1, \dots, w_n$ chosen, choose

w_{n+1} = a point in $\mathrm{span}(v_1, \dots, v_{n+1})$
with $\|w_{n+1}\| = 1$ and
$$1 = \inf_{w \in \mathrm{span}(w_1, \dots, w_n)} \|w_{n+1} - w\|$$
(this exists by problem 4 and the fact that $\mathrm{span}(v_1, \dots, v_{n+1})$ is larger than $\mathrm{span}(w_1, \dots, w_n) = \mathrm{span}(v_1, \dots, v_n)$).

6. If V is infinite dimensional, then $\exists$ a set $v_1, v_2, v_3 \dots$ linearly independent. (Proof: Choose $v_1 \neq 0$. Then choose $v_2 \ni v_1, v_2$ lin. ind set — this is possible since v_1 cannot generate V. Inductively, given $v_1, \dots, v_n$, choose $v_{n+1} \notin \mathrm{span}(v_1, \dots, v_n)$ possible since if $\mathrm{span}(v_1, \dots, v_n) = V$, V is finite dimensional).
But prob. 5, $\exists$ $w_1, w_2 \dots \ni \|w_i\| = 1$ & $d(w_{i+j}, w_i) \geq 1$, $j > 0$.* The sequence $\{w_i\}$ has no convergent subsequence so the unit ball in V is not compact.
Other direction of implication is problem 1.

* since
$$d(w_{i+j}, w_i) \geq d(w_{i+j}, \mathrm{span}(w_1 \dots, w_{i+j-1})) \geq 1.$$

7. Suppose $a_1, \ldots, a_N$ do not span V.
By problem 5, $\exists\ x \ni \|x\| = 1$ and
$\|x - w\| \geq 1$ for all $w \in \operatorname{span}(a_1, \ldots, a_N)$
Clearly $x \notin \bigcup_{1}^{N} B(a_j, \frac{1}{2})$, since
each $a_j \in \operatorname{span}(a_1, \ldots a_N)$, $j = 1, \ldots, N$. $\square$