Homework V: Due Friday, May 22, 2009

1. Prove that if $(V, \| \ \|)$ is a finite-dimensional normed vector space, then the unit ball around $\vec{0}$ $\left(= \{v \in V : \|v\| \leq 1\}\right)$ is compact in the metric on V determined by $\| \ \|$ $\left(d(v,w) = \|v - w\|\right)$.

2. Suppose $(V, \| \ \|)$ is a (not necessarily finite-dimensional) normed vector space and that W is a finite-dimensional subspace. Prove that, if $x \in V$, then $\exists$ $w_x \in W$ such that

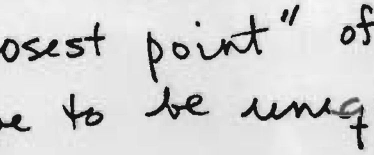
$$\|x - w_x\| = \inf_{w \in W} \|x - w\|, \text{ i.e.,}$$

w_x is a "closest point" of W to x.

3. Does w_x have to be unique? Prove or give an example.

4. Suppose $(V, \| \ \|)$ is a (not necessarily finite dimensional) normed vector space and W is a finite dimensional subspace. Prove: For each $x \in V$, $\exists$ $\hat{w}_x \in \operatorname{span}(W, x)$ such that $\|\hat{w}_x\| = 1$ and $\inf_{w \in W} \|w - \hat{w}_x\| = 1$

[Intuitively, this is like w_x being "pendicular" to W, even though perpendicularity as such does not make sense in V.



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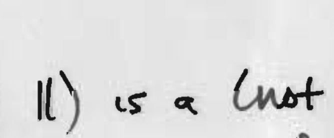
as opposed to



Suggestion: Think about $x - w_x$ from problem 2

5. Suppose V is an infinite dimensional normed vector space with norm $\| \ \|$ and that $v_1, v_2, v_3 \ldots$ is a linearly independent set of vectors in V. Set $W_n = \mathrm{span}(v_1, \ldots, v_n)$. Prove inductively that there is a set of vectors $w_1, w_2, w_3 \ldots$ such that

(i) $\|w_i\| = 1$ for all i

(ii) $W_n = \mathrm{span}(w_1 \ldots, w_n)$, $n = 1, 2, 3 \ldots$

(iii) $\inf_{w \in W_i} \|w - w_{i+1}\| = 1$.

[Suggestion: Use problem 4]

6. Use problems 1 & 5 to prove:

If V is a normed vector space with norm $\| \ \|$, then V is finite-dimensional if and only if the unit ball around $\vec{0}$ $(= \{v : \|v\| \leq 1\})$ is compact.

7. Prove ($(V, \| \ \|)$ a normed vector space as usual) that if $v_1, \ldots, v_n$ is a finite set such that $\{v : \|v\| \leq 1\} \subset \bigcup_{i=1}^{n} B(v_i, \frac{1}{2})$, then $\mathrm{span}(v_1, \ldots, v_n) = V$.

[Suggestion: Use problem 4].