Solutions of Homework IV

1. Suppose X is not connected, so that $X = U \cup V$, U, V open, $U \neq \phi$, $V \neq \phi$, $U \cap V = \phi$. Choose $p \in U$, $q \in V$. If $\gamma : [0,1] \to X$ is continuous with $\gamma(0) = p$, $\gamma(q) = 1$ i.e. γ is an arc (or path) from p to q, then $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are open, nonempty subsets of $[0,1]$ (since $0 \in \gamma^{-1}(U)$ and $1 \in \gamma^{-1}(V)$) with ϕ intersection and union $= [0,1]$. This contradicts the connectedness of $[0,1]$. $\square$

2. Given $\varepsilon > 0$, and $f \in C([0,1])$, choose, by uniform continuity of f and N such that $d(x,y) \leq 1/N$, $x, y \in [0,1] \Rightarrow |f(x) - f(y)| < \varepsilon/10$.
Consider the piecewise linear function F with $r_1 = 0$, $r_2 = \frac{1}{N}$, $r_3 = \frac{2}{N}, \ldots, r_{N+1} = 1$ and value at $=$ a_k where $k = 0, \ldots N$ and a_k is rational and $|f(k/N) - a_k| < \varepsilon/10$.

Then if $x \in [\frac{k}{N}, \frac{k+1}{N}]$, $|f(x) - F(x)| < \varepsilon$ because:

(1) f is within $\varepsilon/10$ of $f(\frac{k}{N})$ on $[\frac{k}{N}, \frac{k+1}{N}]$

(2) $F(\frac{k}{N})$ is within $\frac{\varepsilon}{10}$ of $f(\frac{k}{N})$; and $F(\frac{k+1}{N})$ is within $\frac{\varepsilon}{10}$ of $f(\frac{k+1}{N})$ and hence is within $\frac{\varepsilon}{5}$ of $f(\frac{k}{N})$.

(3) $F(x)$ is between $F(\frac{k}{N})$ and $F(\frac{k+1}{N})$ and hence is with $\frac{2\varepsilon}{5}$ of $f(\frac{k}{N})$.

So $f(x)$ is within $\varepsilon/10$ of $f(\frac{k}{N})$ while $F(x)$ is within $2\varepsilon/5$ of $f(k/N)$. Thus
$|f(x) - F(x)| \leq \frac{\varepsilon}{10} + \frac{2\varepsilon}{5} < \varepsilon$.

So density of rational piecewise linear functions in $C([0,1])$ in sup norm follows.

Note that set of these is countable, since for each fixed $l = 2, 3 \dots$, there are $l-1$ choices of $r_2 \dots, r_{l-1}$ and l choices of values $a_1 \dots, a_l$ (values at $r_1, \dots, r_l$)

So for l fixed, there are countably many. But all rational piecewise linear = union over all $l = 2, 3 \dots$ of these countable sets. hence set of all of them is countable.

So $C([0,1])$ is separable

3. It is elementary to check that sup norm is a metric. Only nontrivial point is Δ inequality. But for each $x_0 \in [0,1]$

$$|f(x_0) - h(x_0)| \le |f(x_0) - g(x_0)| + |g(x_0) - h(x_0)|$$
$$\le \sup_{x \in [0,1]} |f(x) - g(x)| + \sup_{x \in [0,1]} |g(x) - h(x)|$$

So

$$|f(x_0) - h(x_0)| \le d(f,g) + d(g,h)$$

So

$$d(f,h) = \sup_{x_0 \in [0,1]} |f(x_0) - g(x_0)| \le d(f,g) + d(g,h).$$

For nonseparable, recall from class that it suffices to exhibit an uncountable set A such that $a, b \in A \Rightarrow d(a,b) \geq 1$. (Reason: Separable $\Rightarrow$ every open cover of A has countable subcover but with A as described $B(a, \frac{1}{2})$ $a \in A$ is open cover with no countable subcover).
Choose $A = \{\chi_\lambda : \lambda \in [0,1]\}$
where $\chi_\lambda : [0,1] \to \mathbb{R}$ is defined by
$\chi_\lambda(x) = 0$ if $x \neq \lambda$, $\chi_\lambda(\lambda) = 1$. $\square$

4. The map $f \in C([0,1])$, sup norm metric
$\longrightarrow$ $f \in C([0,1])$, L^2 norm metric
is distance nonincreasing
(L_2 norm of $f-g$) $\leq$ sup norm of $f-g$
(elementary integral estimates).
So a set that is dense in $C([0,1])$ in sup norm is necessarily dense in $C([0,1])$ in L^2 norm: If S is ε-dense $\varepsilon > 0$ in sup norm then S is ε-dense in L^2 norm. And density is same as ε-dense for all $\varepsilon > 0$. Thus result of problem 2 implies rational piecewise linear function are L^2 dense in $C([0,1])$.

5. Note that for all $a, b \in \mathbb{R}$, $(a-b)^2 \leq 2(a^2+b^2)$ because $2(a^2+b^2) - (a-b)^2 = a^2+b^2+2ab$
and $|2ab| \leq a^2+b^2$ because
$a^2+b^2 - |2ab| = (|a|-|b|)^2$.

So for each N, $\sum_1^N (x_i - y_i)^2$

$$\leq 2\left(\sum_1^N x_i^2\right) + 2\left(\sum_1^N y_i^2\right).$$

If RHS has a finite limit as $N \to +\infty$ (i.e. if each Σ does) then so does LHS so $\sum_1^{+\infty} (x_i - y_i)^2 < +\infty$.

Also by triangle inequality for $\mathbb{R}^N$

$$\left(\sum_1^N (x_i - y_i)^2\right)^{\frac{1}{2}} \leq \left(\sum_1^N x_i^2\right)^{\frac{1}{2}} + \left(\sum_1^N y_i^2\right)^{\frac{1}{2}}$$

Letting $N \to +\infty$ gives

$$\left(\sum_1^{+\infty} (x_i - y_i)^2\right)^{\frac{1}{2}} \leq \left(\sum_1^{+\infty} x_i^2\right)^{\frac{1}{2}} + \left(\sum_1^{+\infty} y_i^2\right)^{\frac{1}{2}}$$

so

$$\|\vec{x} - \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|.$$

(We could have used this also to prove $\sum_1^{+\infty}(x_i - y_i)^2$ was finite if we had wanted to!)

So metric space $\Delta \leq$ follows

6. If $\sum_{i=1}^{+\infty} x_i^2 < +\infty$, then, given $\varepsilon > 0$,

there is an N such that $\|(x_1, \dots x_N, \dots) - (x_1, \dots, x_N, 0, \dots 0)\|$

$< \varepsilon/2$ since this is same as $\sum_{i=N+1}^{+\infty} x_i^2 < \frac{\varepsilon^2}{4}$ which is true for N large enough (by convergence.

But $\exists$ rational numbers $a_1, \ldots, a_N \ni \| (x_1, \ldots, x_N, 0 \ldots 0)$

$- (a_1, \ldots, a_N, 0 \ldots 0) \| < \varepsilon/2$ by

density of rational numbers in $\mathbb{R}$. Thus
for each $\vec{x} = (x_1, \ldots, x_n, \ldots)$ in l^2

$\exists$ a rational-value eventually 0 sequence
$(a_1, \ldots, a_N, 0 \ldots 0)$ that is ε-close to
$\vec{x}$. Since set of eventually 0
rational-value sequences is countable,
we have a countable dense set in l^2. $\square$

7. If $\sum f_\lambda^2(\lambda_i) \leq M$ for all finite sums,
for each such finite sum and $\varepsilon > 0$
number of λ_i with $|f(\lambda_i)| > \varepsilon \quad \leq M/\varepsilon^2$

λ_i's are points of X here

(otherwise, if there are $> M/\varepsilon^2$ such λ_i,
$\sum$ over those would be $> (M/\varepsilon^2)\varepsilon^2 = M$)
So total number of λ with $f(\lambda) > \varepsilon \quad \leq M/\varepsilon^2$.
$\Lambda_0 = \{\lambda \in \Lambda : |f(\lambda)| > 0\} = \bigcup_{n=1}^{\infty} \{\lambda : |f(\lambda)| > \frac{1}{n}\}$

So Λ_0 is a countable union of finite sets,
hence a countable set itself. $\square$

8. Suppose $f \ni \sum f^2(\lambda_i) \leq M_1$ and $g \ni$
$\sum g^2(\lambda_j) \leq M_2$, both inequalities for all finite
sums. Let $\Lambda_0 = \{\lambda \in X : f(\lambda) \neq 0\}$ and
$\Theta_0 = \{\lambda \in X : g(\lambda) \neq 0\}$. Then Λ_0 and Θ_0 are
countable by problem 7. So $\Lambda_0 \cup \Theta_0$ is countable.

Then f, g are elements of l_2 of $\Lambda_0 \cup \Theta_0$

In the obvious sense, namely let $\Lambda_0 \wedge \Theta_0 =$
$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ldots$ some fixed ordering
and associate to f the sequence

$$f(\lambda_1), f(\lambda_2) \ldots,$$

and to g

$$g(\lambda_1), g(\lambda_2) \ldots .$$

Then as in problem 6, l_2-norm of $f-g$ is defined. If h is also in $l_2(X)$ then $\Lambda_0 \cup \Theta_0 \cup \Delta_0$, $\Delta_0 = \{\lambda \in X : h(\lambda) \neq 0\}$
So again can think of l_2 as in problem 6 except now ordering $\Lambda_0 \cup \Theta_0 \cup \Delta_0$ to get triangle inequality — because it works for l_2 in problem 6 sense.

9. Same general technique as problem 3(b):
For each $\lambda \in X$, let $\chi_\lambda : X \to \mathbb{R}$ be

$$\chi_\lambda(x) = 0 \quad x \in X,\ x \neq \lambda \qquad \chi_\lambda(\lambda) = 1.$$

Then l_2-distance from χ_{λ_1} to χ_{λ_2}

$= \sqrt{2}$ if $\lambda_1 \neq \lambda_2$. So This is an countable set of points all ≥ 1 from each other $\Rightarrow$ $l_2(X)$ not separable.