Homework IV: Due Friday, May 8, 2009

1. A metric space X is arcwise connected if for each $p, q \in X$, $\exists\ \gamma : [0,1] \to X$ γ continuous and $\gamma(0) = p$, $\gamma(1) = q$. Prove that if X is arcwise connected, it is connected.

2. Let P = the set of functions $f : [0,1] \to \mathbb{R}$ such that for some rational numbers $r_1, \dots r_\ell$, $0 = r_1 < r_2 < \dots < r_\ell = 1$, $f(r_i)$ is rational and $f|[r_i, r_{i+1}]$ is linear for $i = 1, \dots, \ell - 1$.
   (a) Prove that P is countable
   (b) Prove that P is dense in $C([0,1])$ so that $C([0,1])$ is separable, both in the $d(f,g) = \max |f(x) - g(x)|$ metric

3. Let $B([0,1])$ be the set of not necessarily continuous but bounded function f on $[0,1]$ (i.e. For each $f \in B([0,1])$, $\exists\ M > 0$ such that $|f(x)| \leq M$ for all $x \in [0,1]$) with metric $d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$
   (a) Prove this is a metric
   (b) Show that $B([0,1])$ is not separable

4. Deduce from problem 2 that $C([0,1])$ with the metric

$$d(f,g) = \left(\int_0^1 (f(x)-g(x))^2\, dx \right)^{\frac{1}{2}}$$

is separable.

5. Let l_2 = the set of all $\mathbb{R}$-valued sequences $\vec{x} = (x_1, x_2, x_3 \ldots)$ such that $\sum_1^\infty x_i^2 < +\infty$.

Show that setting $d(\vec{x}, \vec{y}) = \left(\sum (x_i - y_i)^2\right)^{\frac{1}{2}}$ defines a metric on l_2.
[This takes some work!]

6. Prove that l_2 of problem 5 is separable.

7. Show that if X is an uncountable set and if $f: X \to \mathbb{R}$ is a function such that for some $M > 0$, $\sum_{i=1}^{l} f(x_i)^2 \leq M$

for all finite sets $x_1, \ldots, x_l \in X$ (l arbitrary), then

$$\{x \in X : f(x) \neq 0\} \text{ is countable.}$$

8. Let $l_2(X)$, X uncountable, be the set of functions f satisfying the condition of problem 7.
(a) Show $d(f,g) = \sup\left(\sum (f(x_i) - g(x_i))^2\right)^{\frac{1}{2}}$

where the sum is over finite subsets of X and the sup is over all finite subsets, defines a metric on $l_2(X)$.

9. Show that $l_2(X)$ is not separable if X is uncountable.