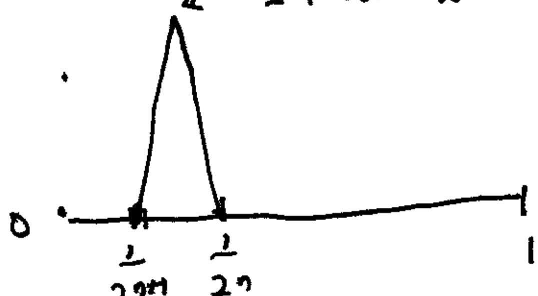
Solutions of Homework III

1. Let $f_n : [0,1] \to \mathbb{R}$ be the function with graph as shown

$= 1$ at $x = \frac{1}{2}\left(\frac{1}{2^{n+1}} + \frac{1}{2^n}\right)$



Then f_n's are continuous. Also $d(\vec{0}, f_n) = 1$.

$d(f_n, f_m) \le 1$ clearly

And $d(f_n, f_m) = 1$ since $f_m\left(\frac{1}{2}\left(\frac{1}{2^{n+1}} + \frac{1}{2^n}\right)\right) = 0$

while $f_n\left(\frac{1}{2}\left(\frac{1}{2^{n+1}} + \frac{1}{2^n}\right)\right) = 0$ but

if $m \ne n$ (because $\left[\frac{1}{2^{m+1}}, \frac{1}{2^m}\right]$ and $\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$ are disjoint except possibly for an endpoint.

So $\{f_n\}$ is a sequence in $\{f : d(\vec{0}, f) \le 1\}$ with no convergent subsequence

2. $\int_0^1 \sin^2 2\pi n x \, dx = \frac{1}{2}$ while, if $m \ne n$,

$$\int_0^1 (\sin 2\pi n x - \sin 2\pi m x)^2 = \int_0^1 \sin^2 2\pi n x + \int_0^1 \sin^2 2\pi m x - 2\int \sin 2\pi m x \sin 2\pi n x \, dx.$$

Now $\int_0^1 \sin 2\pi n x \sin 2\pi m x \, dx = 0$ if $m \ne n$

(use $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$)

So $\int_0^1 (\sin 2\pi n x - \sin 2\pi m x)^2 = 1$ if $m \ne n$.

So $f_n(x) = \sin 2\pi n x$ is a sequence with $d(\vec{0}, f_n) = \frac{1}{2}$ but not convergent subsequence.

3. (a) If $x \in C$, there is nothing to prove. ②

So suppose $x \notin C$ and let $d_0 = \inf_{y \in C} d(x,y)$.

But definition of inf, $\exists\ y_i \in C$ such that $\lim d(y_i, x) = d_0$. For all i sufficiently large, $y_i \in \{y \in X : d(x,y) \leq d_0 + 1\} \cap C$.

This set is compact since $\{\ \}$ set is closed & bounded while C is closed so $\{\ \} \cap C$ is closed & bounded.

So $\exists\ y_{i_k} \to y_0$ (y_{i_k} a subsequence of the $\{y_i\}$ sequence, i from some i_0 onwards), $y_0 \in C$ (since C is closed)

Then $d(x, y_0) = \lim d(x, y_{i_k}) = d_0$

where we have used $d(x, \lim y_{i_k}) = \lim d(x, y_{i_k})$

(which follows from the triangle inequality).

4. $d(C_1, C_2) = d(C_2, C_1)$ and $d(C_1, C_2) \geq 0$ are obvious from the definitions as is $d(C, C) = 0$.

To check $d(C_1, C_2) = 0 \Rightarrow C_1 = C_2$, note that, given $\varepsilon > 0$, $d(C_1, C_2) = 0$ & $x \in C_1 \Rightarrow \exists\ y \in C_2 \ni d(x,y) < \varepsilon$.

So if $d(C_1, C_2) = 0$ & $x \in C_1$ then x is adherent to C_2, hence in C_2. Similarly $x \in C_2$ and $d(C_1, C_2) = 0 \Rightarrow x \in C_1$. So $d(C_1, C_2) = 0 \Rightarrow C_1 = C_2$.

(note closedness is essential: for sets that are just bounded, this would not work).

→ 3(b) No. Example: $X = \mathbb{R} - \{0\}$, $C = \{x \in \mathbb{R} - \{0\} : x < 0\}$

$x = 1$. 3(c) Yes: Same argument as for (a) without the need for intersecting with $\{\ \ \leq \inf + 1\}$ set.

4(continued). To prove the triangle inequality, suppose $d(C_1,C_2)=\varepsilon_1$ and $d(C_2,C_3)=\varepsilon_2$. If $r>\varepsilon_1$ and $s>\varepsilon_2$ and $x\in C_1$, then $\exists\, y\in C_2 \ni d(x,y)<r$ and $z\in C_3 \ni d(y,z)<s$. Hence $d(x,z)< r+s$. Similarly for any $z\in C_3$, $\exists\, x\in C_1$ such that $d(z,x)<r+s$. Thus $d(C_1,C_3)\le r+\varepsilon$. Since this is true for all $r>\varepsilon_1$ and $s>\varepsilon_2$, it follows that $d(C_1,C_3)\le \varepsilon_1+\varepsilon_2$.

5. The space is complete (of all compact subsets in the specified metric). To prove this, note first that if $C_1,C_2,C_3,\dots$ is a Cauchy sequence of compact sets then $\bigcup C_i$ is bounded. To see this, choose i_0 so large that $d(C_{i_0},C_j)<\frac{1}{2}$ if $i>i_0$. Then

$$\bigcup C_k \subset C_1\cup\dots\cup C_{i_0-1}\cup\{x\in\mathbb{R}^2: d(x,C_{i_0})\le 1\}$$

and this set is clearly bounded.

Continuing to suppose that $C_1,C_2,C_3\dots$ is a Cauchy sequence of compact sets, we look for a candidate for a limit of the sequence as follows: Let $C_0=$ the set of all points $x\in\mathbb{R}^2$ such that for some subsequence i_j $j=1,2,3\dots$ of the positive integers there is a sequence $x_{i_j}\in C_{i_j}$ such that $x_{i_j}\to x$. Then:

C_0 is bounded, since $\bigcup C_j$ is bounded.
And C_0 is closed: If y_0 is a limit of a seq of such points y_j, then there is a (sub)sequence of the sort described obtainable by choosing for $n = 1, 2, 3 ..$ points $\frac{1}{n}$ close to y_n in set C_{i_n} where i_n is a subsequence of $1, 2, 3 ....$ So C_0 is compact.

It remains to see that $d(C_i, C_0) \to 0$ as $i \to +\infty$. For this, we need to show that (a) given $\varepsilon > 0$, and $x \in C_0$, there exists, for all sufficiently large i, a point $y_i \in C_i$ with $d(x, y_i) < \varepsilon$. and that (b) again given $\varepsilon > 0$, for all large enough i and $y_i \in C_i$ there is a point $x_i \in C_0 \ni d(x_i, y_i) < \varepsilon$.

For the first part, choose N_0 such that $i, j > N_0 \Rightarrow d(C_i, C_j) < \varepsilon/10$. Then with x_{i_j} being the subsequence (in C_{i_j}) converging to x (which exists by definition of C_0), choose a fixed j so large that $i_j > N_0$. and so large that $d(x_{i_j}, x) < \varepsilon/10$.
Then for all $i > N_0$, $\exists\, y_i \ni d(y_i, y_{i_j}) < \varepsilon/10$ while $d(y_{i_j}, x) < \varepsilon/10$ so that $d(y_i, x) < \varepsilon$.
The second half of what what we need to show, part (b) is a little trickier: Suppose there is some $\varepsilon > 0$ for which it does not work. Then there

is a subsequence i_j and $y_{i_j} \in C_{i_j}$ such that every point of C_0 has distance $\geq \varepsilon$ from y_{i_j}. Now the sequence $\{y_{i_j}\}$ is bounded (since it lies in $\bigcup C_i$). So it has a convergent subsequence, say $\{y_{i_{j_k}}\}$. This subsequence's limit is in C_0 by definition of C_0. But the y_{i_j} all had distance $\geq \varepsilon$ from C_0. This is a contradiction!

(Here we are using dis(pt, compact set) = $\inf_{z \in \text{compact set}} d(pt, z)$ as in problem 3). □

It is worth noting that we did not use much about $\mathbb{R}^2$. The whole thing could have been done if we just supposed, for instance, that all sets C were closed subsets of a fixed compact metric space X.

6.(b) Because the ε- balls cover

$$(\# S_\varepsilon)\,(\text{area of an } \varepsilon\text{-ball}) \geq 4\pi \quad (= \text{area of unit sphere})$$

Since $\varepsilon/2$ - balls are disjoint

$$(\# S_\varepsilon)\,(\text{area of an } \varepsilon/2\text{-ball}) \leq 4\pi$$

Given that area of an ε-ball is $\geq A_1 \varepsilon^2$ and $\leq A_2 \varepsilon^2$

one gets

$$(\# S_\varepsilon) \leq \frac{4\pi}{\text{area of } \varepsilon/2 \text{ ball}} \leq \frac{4\pi}{A_1 (\frac{\varepsilon}{2})^2} \leq \frac{16\pi}{A_1} \varepsilon^{-2}$$

while

$$(\# S_\varepsilon) \geq \frac{4\pi}{\text{area of } \varepsilon \text{ ball}} \geq \frac{4\pi}{A_2 \varepsilon^2} = \frac{4\pi}{A_2} \varepsilon^{-2}.$$

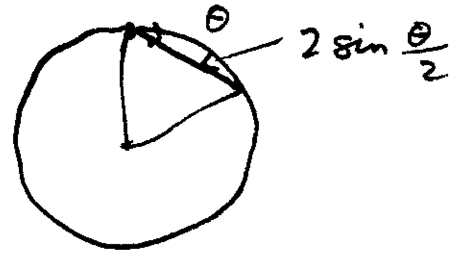
(a) Metric space is clear except $\Delta \leq$: that follows since great circle distance $\leq$ length of any other path on sphere so

$$\ell(\text{great circle from } x \text{ to } y) + \ell(\text{great circle } y \text{ to } z)) \geq \ell(\text{great circle}(x,z))$$

where all great circles are shortest arcs.

So $\Delta \leq$ follows.

Compact follows because great circle distance is small $\Longleftrightarrow$ $\mathbb{R}^3$ distance is small since $\mathbb{R}^3$ distance $= 2 \sin \frac{\text{great circle dis}}{2}$



So open sets for great circle distance are same as open sets for $\mathbb{R}^3$ distance and sphere is compact in $\mathbb{R}^3$-induced topology (closed & bounded in $\mathbb{R}^3$)