Homework III. Due Friday, April 24, 2009

1. Let $C([0,1])$ = the set of continuous functions, $\mathbb{R}$-valued, on $[0,1]$. Define the metric d on $C([0,1])$ by $d(f,g) = \max_{x\in[0,1]} |f(x)-g(x)|$. Prove that the closed unit ball $\{f : d(0,f) \le 1\}$ around the 0-function is not compact.

2. Same as problem 1 with metric $d(f,g) = \left(\int_0^1 (f-g)^2\,dx\right)^{\frac{1}{2}}$

[Suggestion: Think about the sequence $f_n(x) = \sin(2\pi n x)$] noting that $\int_0^1 \sin(2\pi n x)\sin(2\pi m x)\,dx = 0$ if $m \neq n$.

3(a) Prove that if C is a closed subset of $\mathbb{R}^n$ and x is a point of $\mathbb{R}^n$ then there is a point $y_0 \in C$ such that

$$d(x, y_0) = \inf_{y\in C} d(x,y)$$

(inf = greatest lower bound). (d = usual Pythagorean distance)

(b) Is (a) always true for a closed set C and point x in a general metric space X?

(c) Is (a) always true for a compact set C and point x in a general metric space X?

4. Define a metric d on the set of all compact subsets of $\mathbb{R}^2$ by

$d(C_1, C_2)$ = greatest lower bound of the set of $r \in \mathbb{R}$ such that
$$C_1 \subset \bigcup_{p \in C_2} B(p,r) \quad \text{and} \quad C_2 \subset \bigcup_{p \in C_1} B(p,r)$$

Show that this is a metric, that is, that it satisfies $d(C_1, C_2) \geq 0$ and $= 0 \Leftrightarrow C_1 = C_2$, $d(C_1, C_2) = d(C_2, C_1)$, and d satisfies the triangle inequality.

* 5. Is the metric you worked with in problem 4 complete?
[cf., prob. 8, p. 25 of G&G]

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6. Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Put the metric on S^2 determined by "great circle distance" (the usual idea of how far one place on earth is from another) (a) Then S^2 is a compact metric space: prove this
(b) Let S_ε, $\varepsilon > 0$, be an ε-net in S^2, i.e., a (finite) set of pts such that $p, q \in S_\varepsilon \Rightarrow d(p,q) \geq \varepsilon$ and $\bigcup_{p \in S_\varepsilon} B(p,\varepsilon) = S^2$. Let $\# S_\varepsilon$ = the number of points in S_ε. Prove that there are constants

C_1, C_2, $0 < C_1 < C_2$ such that

$$C_1 \varepsilon^{-2} \leq \# S_\varepsilon \leq C_2 \varepsilon^{-2}$$

for all $\varepsilon \in (0,1)$.

[Suggestion: $\bigcup_{p \in S_\varepsilon} B(p,\varepsilon) = S^2$ so sum of areas of $B(p,\varepsilon)$, $p \in S_\varepsilon$ $= (\# S_\varepsilon)$ (area of ε-disc) $\geq$ area of S^2 $= 4\pi$. But $B(p, \varepsilon/2)$, $p \in S_\varepsilon$ are disjoint pairwise so that . . area statement . . .

You may <u>assume</u> (as is true) that area of radius-ε disc is $\geq A_1 \varepsilon^2$ and $\leq A_2 \varepsilon^2$ for some constants $A_1, A_2 > 0$ and all $\varepsilon \in (0,1)$.]