Math 121 Howework II

1.(a) First note that any rational ball $B(p, r)$ is uniquely determined by the point $p \in \mathbb{Q}^n$ and $r \in \mathbb{Q}_{>0}$ (this means positive rationals). Hence we have an injection into $\mathbb{Q}^{n+1}$. So it suffices to show that $\mathbb{Q}^n$ is countable for any $n \geq 1$. This is true for $n = 1$. Assume true for some $n \geq 1$. Now for any $q \in \mathbb{Q}$ the set $\{(q,p) : p \in \mathbb{Q}^n\}$ is in bijection with $\mathbb{Q}^n$, hence countable by assumption. Then

$$\mathbb{Q}^n = \bigcup_{q\in\mathbb{Q}} \{(q,p) : p \in \mathbb{Q}^n\}$$

is a countable union of countable sets, hence countable. By induction the result follows.

1.(b) Let $\mathcal{B}_U = \{B(p,r) : p \in \mathbb{Q}^n,\ r \in \mathbb{Q}_{>0} \text{ and } B(p,r) \subseteq U\}$. Since each $B(p,r) \in \mathcal{B}_U$ is contained in U it is clear that

$$\bigcup_{B(p,r)\in\mathcal{B}_U} B(p,r) \subseteq U.$$

Now take any $x \in U$. Since U is open there is some $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq U$. Be density of the rationals we know there is some $p \in \mathbb{Q}^n$ such that $||x - y|| < \varepsilon/3$. By density of the rationals, again, we know that there is some $r \in \mathbb{Q}$ with $\varepsilon/3 < r < \varepsilon/2$. Then note that $x \in B(p,r)$ and for any $y \in B(p,r)$ we have

$$||y - x|| \leq ||y - p|| + ||p - x|| < \varepsilon/2 + \varepsilon/3 < \varepsilon,$$

so $B(p,r) \subset U$. It follows that

$$U \subseteq \bigcup_{B(p,r)\in\mathcal{B}_U} B(p,r).$$

2. For each $\lambda \in \Lambda$ let $\mathcal{B}_\lambda = \{B(p,r) : p \in \mathbb{Q}^n,\ r \in \mathbb{Q}_{>0} \text{ and } B(p,r) \subseteq U_\lambda\}$ and let $\mathcal{B} = \cup_{\lambda\in\Lambda}\mathcal{B}_\lambda$. By question 1.(a) we know that $U_\lambda = \cup_{B(p,r)\in\mathcal{B}_\lambda} B(p,r)$, hence

$$S \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda = \bigcup_{\lambda\in\Lambda} \left(\bigcup_{B(p,r)\in\mathcal{B}_\lambda} B(p,r) \right) = \bigcup_{B(p,r)\in\mathcal{B}} B(p,r).$$

By question 1.(a) we see that $\mathcal{B}$ is countable (being a subset of a countable set). Write $\mathcal{B} = \{B(p_i, r_i) : i \geq 1\}$. For each $i \geq 1$ take $\lambda_i \in \Lambda$ such that $B(p_i, r_i) \in \mathcal{B}_{\lambda_i}$, i.e. $B(p_i, r_i) \subseteq U_{\lambda_i}$. Then

$$S \subseteq \bigcup_{B(p,r)\in\mathcal{B}} B(p,r) = \bigcup_{i=1}^{\infty} B(p_i, r_i) \subseteq \bigcup_{i=1}^{\infty} U_{\lambda_i}.$$

3. Assume S does not contain any condensation point of itself. Then for every $x \in S$ there is some $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \cap S$ is at most countable. Now

$$S \subseteq \bigcup_{x\in S} B(x, \varepsilon_x)$$

and by question 2 we can find some countable subcollection $\{x_i : i \geq 1\}$ such that

$$S \subseteq \bigcup_{i=1}^{\infty} B(x_i, \varepsilon_{x_i}).$$

Then

$$S \subseteq \left(\bigcup_{i=1}^{\infty} B(x_i, \varepsilon_{x_i}) \right) \cap S = \bigcup_{i=1}^{\infty} (B(x_i, \varepsilon_i) \cap S),$$

which is a countable union of at most countable sets, hence countable.

4.(a) Let $C \subseteq R^n$ be the set of condensation points of S and take $x \in \mathbb{R}^n \setminus C$. Since x is not a condensation point of S there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \cap S$ is at most countable. Take $y \in B(x, \varepsilon)$ and set $\delta = \varepsilon - ||x - y||$. Then we have that $B(y, \delta) \subseteq B(x, \varepsilon)$ and $B(y, \delta) \cap S \subseteq B(x, \varepsilon) \cap S$. Thus $B(y, \delta) \cap S$ is at most countable, so $y \in \mathbb{R}^n \setminus C$, that is $B(x, \varepsilon) \subseteq \mathbb{R}^n \setminus C$. It follows that C is closed.

4.(b) Let C denote the condensation points of S and let $p \in C$. For each $n \geq 1$ we see that $B(p, 1/n) \cap S$ is uncountable, hence $(B(p, 1/n) \cap S) \setminus \{p\}$ is uncountable. By question 3 there exists some $x_n \in B(p, 1/n)$ with $x_n \neq p$ and x_n a condensation point of S. Since $||x_n - p|| < 1/n$ for all $n \geq 1$ we see that the sequence $(x_n)_{n \geq 1}$ converges to p.

5.(a) Let C denote the set of condensation points of S. Assume that $S \setminus C$ is uncountable. Then, by question 3, $S \setminus C$ has a condensation point belonging to $S \setminus C$, i.e. there is some $p \in S \setminus C$ such that for any $\varepsilon > 0$, $B(p, \varepsilon) \cap (S \setminus C)$ is uncountable. Hence, for every $\varepsilon > 0$, $B(p, \varepsilon) \cap S$ is uncountable and p is a condensation point of S. This is a contradiction.

5.(b) Let C denote the set of condensation points of S. By the definition of condensation points of S we see that any condensation point of S is adherent to S. Since S is closed we conclude $C \subseteq S$ (note this is not true for arbitrary S, for example the set of condensation points of the open interval $(0, 1)$ in $\mathbb{R}$ is the closed interval $[0, 1]$). Then write $S = C \cup (S \setminus C)$. By questions 4 we see that C is perfect and by 5.(a) we see that $S \setminus C$ is countable.

6. Let C denote the Cantor set. Then $C = \cap_{i=0}^{\infty} E_n$ where each E_n is a union of 2^n closed intervals each of length 3^{-n} constructed inductively as follows. Let $E_0 = [0, 1]$. Assuming E_n is constructed we construct E_{n+1} by deleting from E_n the open interval $(a_i + 3^{-(n+1)}, b_i - 3^{-(n+1)})$ from each of the 2^n closed intervals $[a_i, b_i]$ in E_n. Note that since each E_n is a finite union of closed intervals it is closed. Then C is closed as it is the intersection of a collection of closed sets. It remains to show C contains no isolated points.

Take any $x \in C$. Then $x \in E_n$, as above, for each $n \geq 1$. Hence for each $n \geq 1$ we see that x lies in some closed interval $[a_{n,i}, b_{n,i}]$ of length 3^{-n} and for each $n \geq 1$, $a_{n,i}, b_{n,i} \in C$. So for each $n \geq 1$ set $x_n = a_{n,i}$ unless $x = a_{n,i}$, in which case we set $x_n = b_{n,i}$. Then we have, for all $n \geq 1$, that $x_n \in C$, $x_n \neq x$ and $|x - x_n| \leq 3^{-n}$. So $(x_n)_{n \geq 1}$ is a sequence of elements of C, distinct from x and converging to x, so x is not isolated. It follows that C is perfect.

7. Let $S \subseteq \mathbb{R}^n$ be a perfect set. Since any closed subset of a complete metric space is complete we see that S is complete. So it suffices to show that any complete metric space without isolated points in uncountable.

Clearly any finite metric space contains isolated points. Say (X, d) is a countable complete metric space without isolated points. Write $X = \{x_n : n \geq 1\}$. Consider the sets $U_n = X \setminus \{x_n\}$. Since $\{x_n\}$ is closed, U_n is open. Since x_n is not isolated, for all $\varepsilon > 0$, $B(x_n, \varepsilon) \cap U_n \neq \emptyset$, hence U_n is dense in X (any other point not equal to x is already contained in U_n). By the Baire Category Theorem $\cap_{i=1}^{\infty} U_n$ is dense in X. But $\cap_{i=1}^{\infty} U_n = \emptyset$, a contradiction.