Homework II : Mathematics 121, Spring 2009
Due Friday, April 17, 2009

1. A <u>rational</u> <u>ball</u> in $\mathbb{R}^n$ is a set of the form $B(p, r)$, where r is a rational (positive) number and p has rational coordinates, i.e. $p = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ where each $\alpha_i \in \mathbb{Q}$.
   (a) Prove that the collection of all rational balls in $\mathbb{R}^n$ is countable.
   (b) Prove that each open set $U \subset \mathbb{R}^n$ is the union of the rational balls $B(p,r) \subset U$.

2. Prove: If S is a set in $\mathbb{R}^n$ and $S \subset \bigcup_{\lambda \in \Lambda} U_\lambda$, each U_λ open in $\mathbb{R}^n$, then $\exists\ \lambda_1, \lambda_2, \lambda_3 \dots$ such that $S \subset \bigcup_{j=1}^{+\infty} U_{\lambda_j}$.

Slogan: "Every open cover has a countable subcover".
[Suggestion: Let $\mathcal{B}$ = the collection of rational balls $B(p,r)$ such that $B(p,\lambda) \subset U_\lambda$ for some λ. Note that $S \subset \bigcup_{B(p,r) \in \mathcal{B}} B(p,r)$ and that $\mathcal{B}$ is countable].

3. A <u>condensation point</u> of a set S is a point p such that, $\forall \varepsilon > 0$, $B(p,\varepsilon) \cap S$ is uncountable. Prove that every uncountable set S in $\mathbb{R}^n$ has a condensation point p belonging to S.

[Suggestion: If not, $\{B(p,r) : p \in S,\ r>0 \ni S \cap B(p,r)$ is countable$\}$ is an open cover of S. Apply problem 2]

4. Prove: (a) If S is a subset of $\mathbb{R}^n$, the set of condensation points of S is closed in $\mathbb{R}^n$

(b) Every condensation point p of S is a limit of a sequence of condensation point of S each different from p.

Slogan: "The set of condensation points is dense-in-itself": "dense in itself" means ~~to~~ no point is isolated.

[Suggestion: $B(p,\varepsilon) = \{p\} \cup \left(\bigcup_{n=1}^{\infty} \{x : \frac{\varepsilon}{1+n} \le dis(p,x) < \frac{\varepsilon}{n}\}\right)$

so $B(p,\varepsilon) \cap S$ uncountable implies ....]

5. Prove (a) If S is a subset of $\mathbb{R}^n$, then

$S \sim$ (condensation points of S) is countable.

(b) Every ~~set~~ closed set in $\mathbb{R}^n$ is the union of countable set and a "perfect set", where: a set is called "perfect" if it is closed and "dense in itself" ["dense in itself" means no point is isolated], as in problem 4].

6. Prove: The Cantor set is perfect.

7. Prove: Every (nonempty) perfect set in $\mathbb{R}^n$ is uncountable.