Math 121 Howework I

1.(a) We have, for any $(x_1, y_1), (x_2, y_2) \in X \times Y$, since $d_x(x_1, x_2) \geq 0$ and $d_Y(y_1, y_2) \geq 0$, that

$$D((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2) \geq 0.$$

Then $D((x_1, y_1), (x_2, y_2)) = 0$ if and only if $d_X(x_1, x_2) = 0$ and $d_Y(y_1, y_2) = 0$, which happens if and only if $x_1 = x_2$ and $y_1 = y_2$, that is $(x_1, y_1) = (x_2, y_2)$. Also

$$D((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2) = d_X(x_2, x_1) + d_Y(y_2, y_1) = D((x_2, y_2), (x_1, y_1).$$

Now take $(x_3, y_3) \in X \times Y$. Then

$$\begin{aligned} D((x_1, y_1), (x_3, y_3)) &= d_X(x_1, x_3) + d_Y(y_1, y_3) \\ &\leq d_X(x_1, x_2) + d_X(x_2, x_3) + d_Y(y_1, y_2) + d_Y(y_2, y_3) \\ &= D((x_1, y_2), (x_2, y_2)) + D((x_2, y_2), (x_3, y_3)). \end{aligned}$$

And so $(X \times Y, D)$ is a metric space.

1.(b) Let $((x_n, y_n))_{n\geq 1}$ be a sequence in $X \times Y$. Assume that $((x_n, y_n))_{n\geq 1}$ converges to (x, y) in $X \times Y$. Then for any $\varepsilon > 0$ there is some $N \geq 1$ such that for all $n \geq N$ we have $D((x_n, y_n), (x, y)) < \varepsilon$. Hence for any $n \geq N$

$$d_X(x_n, x) \leq d_X(x_n, x) + d_Y(y_n, y) < \varepsilon \quad \text{and} \quad d_Y(y_n, y) \leq d_X(x_n, x) + d_Y(y_n, y) < \varepsilon,$$

so $(x_n)_{n\geq 1}$ converges to x in X and $(y_n)_{n\geq 1}$ converges to y in Y.

Now assume $(x_n)_{n\geq 1}$ converges to x in X and $(y_n)_{n\geq 1}$ converges to y in Y. Fix $\varepsilon > 0$. There is some $N_1 \geq 1$ and $N_2 \geq 1$ such that for all $n \geq N_1$ we have $d_X(x_n, x) < \varepsilon/2$ and for all $n \geq N_2$ we have $d_Y(y_n, y) < \varepsilon/2$. Hence for all $n \geq \max\{N_1, N_2\}$ we have

$$D((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so $((x_n, y_n))_{n\geq 1}$ converges to (x, y) in $X \times Y$.

2. Let $((x_n, y_n))_{n\geq 1}$ be a sequence in $X \times Y$. Let $\sigma_1 : \mathbb{N} \to \mathbb{N}$ be an stictly increasing function such that the subsequence $(x_{\sigma_1(k)})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ converges in X. Now consider the sequence $(y_{\sigma_1(k)})_{k\geq 1}$ in Y. It has a convergent subsequence so let $\sigma_2 : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $(y_{\sigma_2(k)})_{k\geq 1}$ is a convergent subsequence of $(y_{\sigma_1(k)})_{k\geq 1}$. Since $(x_{\sigma_2(k)})_{k\geq 1}$ is a subsequence of $(x_{\sigma_1(k)})_{k\geq 1}$, which is convergent, $(x_{\sigma_2(k)})_{k\geq 1}$ is convergent. By question 1.(b) the subsequence $((x_{\sigma_2(k)}, y_{\sigma_2(k)}))_{k\geq 1}$ of $((x_n, y_n))_{n\geq 1}$ converges.

3. First note that since $d(x, y) \geq 0$ for all $x, y \in X$ we clearly have $\hat{d} = \min\{1, d(x, y)\} \geq 0$ for all $x, y \in X$. Now $\hat{d}(x, y) = 0$ if and only if $d(x, y) = 0$ which happens if and only if $x = y$. Next we see easily that $\hat{d}(x, y) = \min\{1, d(x, y)\} = \min\{1, d(y, x)\} = \hat{d}(y, x)$.

Now take $z \in X$. We want to show $\hat{d}(x, z) \leq \hat{d}(x, y) + \hat{d}(y, z)$. First assume that at least one of $d(x, y) \geq 1$ or $d(y, z) \geq 1$ holds. Then either $\hat{d}(x, y) = 1$ or $\hat{d}(y, z) = 1$ or both and

$$\hat{d}(x, z) = \min\{1, d(x, z)\} \leq 1 \leq \hat{d}(x, y) + \hat{d}(y, z).$$

Now assume that both $d(x, y) < 1$ and $d(y, z) < 1$. Then

$$\hat{d}(x, z) = \min\{1, d(x, z)\} \leq d(x, z) \leq d(x, y) + d(y, z) = \hat{d}(x, y) + \hat{d}(y, z).$$

Hence $(X, \hat{d})$ is a metric space.

4.(a) Denote, for each $i \geq 1$, by $\hat{d}_i$ the metric on X_i given by $\hat{d}_i(x_i, y_i) = \min\{1, d_i(x_i, y_i)\}$. Note that for each $i \geq 1$ and any $x_i, y_i \in X_i$ we have $\hat{d}(x_i, y_i) \leq 1$. Hence the series

$$\sum_{i=1}^{\infty} 2^{-i} \hat{d}(x_i, y_1)$$

converges since the series $\sum_{i=1}^{\infty} 2^{-i}$ converges. So d_π is defined.

Since d_π is a sum of nonnegative terms it is clearly nonnegative. Since each term in the summation defining $d_\pi((x_i),(y_i))$, for $(x_i),(y_i) \in \prod_{i=1}^{\infty} X_i$, is nonnegative, it follows that $d_\pi((x_i),(y_i)) = 0$ if and only if each of these terms is equal to zero, i.e. $\hat{d}(x_i, y_i) = 0$ for all $i \geq 1$. Hence $d_\pi((x_i),(y_i)) = 0$ if and only if $x_i = y_i$ for all $i \geq 1$, i.e. $(x_i) = (y_i)$. Also

$$d_\pi((x_i),(y_i)) = \sum_{i=1}^{\infty} 2^{-i}\hat{d}(x_i, y_i) = \sum_{i=1}^{\infty} 2^{-i}\hat{d}(y_i, x_i) = d_\pi((y_i),(x_i)).$$

Now take $(z_i) \in \prod_{i=1}^{\infty} X_i$. Since each of $\sum_{i=1}^{\infty} 2^{-i}\hat{d}(x_i, y_i)$ and $\sum_{i=1}^{\infty} 2^{-i}\hat{d}(y_i, z_i)$ are convergent we have

$$\begin{aligned} d_\pi((x_i),(z_i)) &= \sum_{i=1}^{\infty} 2^{-i}\hat{d}(x_i, z_i) \\ &\leq \sum_{i=1}^{\infty} 2^{-i}(\hat{d}(x_i, y_i) + \hat{d}(y_i, z_i)) \\ &= \sum_{i=1}^{\infty} 2^{-i}\hat{d}(x_i, y_i) + \sum_{i=1}^{\infty} 2^{-i}\hat{d}(y_i, z_i) \\ &= d_\pi((x_i),(y_i)) + d_\pi((y_i)),(z_i)). \end{aligned}$$

So $(\prod_{i=1}^{\infty} X_i, d_\pi)$ is a metric space.

4.(b) Let $((x_i^j))_{j\geq 1}$ be a sequence of elements in $\prod_{i=1}^{\infty} X_i$, i.e. for each $j \geq 1$, (x_i^j) is an element of $\prod_{i=1}^{\infty} X_i$. First assume that $((x_i^j))_{j\geq 1}$ converges to some (x_i) in $\prod_{i=1}^{\infty} X_i$. Fix arbitrary $n \geq 1$ and $\varepsilon > 0$. Letting $\varepsilon' = \min\{\varepsilon, 1\}$, there is some $N \geq 1$ such that $d_\pi((x_i^j),(x_i)) < 2^{-n}\varepsilon'$ for all $j \geq N$. Then for all $j \geq N$ we have

$$\hat{d}_n(x_n^j, x_n) \leq 2^n \sum_{i=1}^{\infty} \hat{d}_i(x_i^j, x_i) = 2^n d_\pi((x_i^j),(x_i)) < \varepsilon'.$$

Since $\varepsilon' \leq 1$ we get that $\hat{d}_n(x_n^j, x_n) = d_n(x_n^j, x_n) < \varepsilon' \leq \varepsilon$ for all $j \geq N$. Hence each $(x_i^j)_{j\geq 1}$ converges to x_i in X_i.

Now assume that for each $i \geq 1$ the sequence $(x_i^j)_{j\geq 1}$ converges to some x_i in X_i. Fix $\varepsilon > 0$ and take $M \geq 1$ such that $2^{-M} < \varepsilon/2$. For each $1 \leq i \leq M$ there is some N_i such that for all $j \geq N_i$ we have $d_i(x_i^j, x_i) < \varepsilon/(2M)$. Then for any $j \geq \max\{N_1, \ldots, N_M\}$ we have

$$\begin{aligned} d_\pi((x_i^j),(x_i)) &= \sum_{i=0}^{\infty} 2^{-i}\hat{d}(x_i^j, x_i) \\ &= \sum_{i=0}^{M} 2^{-i}\hat{d}_i(x_i^j, x_i) + \sum_{i=M+1}^{\infty} 2^{-i}\hat{d}(x_i^j, x_i) \\ &\leq \sum_{i=0}^{M} d_i(x_i^j, x_i) + \sum_{i=M+1}^{\infty} 2^{-i} \\ &< M(\varepsilon/(2M)) + 2^{-M} \\ &< \varepsilon. \end{aligned}$$

Hence $((x_i^j))_{j\geq 1}$ converges to (x_i) in $\prod_{i=1}^{\infty} X_i$.

5. First we want to show that a sequence $((x_i^j))_{j\geq 1}$ in $\prod_{i=1}^{\infty} X_i$ is Cauchy if and only if each sequence $(x_i^j)_{j\geq 1}$ is Cauchy in X_i.

First assume that $((x_i^j))_{j\geq 1}$ is Cauchy. Fix arbitrary $n \geq 1$ and $\varepsilon > 0$. Letting $\varepsilon' = \min\{\varepsilon, 1\}$, there is some $N \geq 1$ such that $d_\pi((x_i^j),(x_i^k)) < 2^{-n}\varepsilon'$ for all $j,k \geq N$. Then for all $j,k \geq N$ we have

$$\hat{d}_n(x_n^j, x_n^k) \leq 2^n \sum_{i=1}^{\infty} \hat{d}_i(x_i^j, x_i^k) = 2^n d_\pi((x_i^j),(x_i^k)) < \varepsilon'.$$

Since $\varepsilon' \leq 1$ we get that $\hat{d}_n(x_n^j, x_n^k) = d_n(x_n^j, x_n^k) < \varepsilon' \leq \varepsilon$ for all $j,k \geq N$. Hence each $(x_i^j)_{j\geq 1}$ is Cauchy.

Now assume that for each $i \geq 1$ the sequence $(x_i^j)_{j\geq 1}$ is Cauchy. Fix $\varepsilon > 0$ and take $M \geq 1$ such that $2^{-M} < \varepsilon/2$. For each $1 \leq i \leq M$ there is some N_i such that for all $j,k \geq N_i$ we have $d_i(x_i^j, x_i^k) < \varepsilon/(2M)$. Then for any $j,k \geq \max\{N_1,\ldots,N_M\}$ we have

$$\begin{aligned} d_\pi((x_i^j),(x_i^k)) &= \sum_{i=0}^{\infty} 2^{-i}\hat{d}(x_i^j, x_i^k) \\ &= \sum_{i=0}^{M} 2^{-i}\hat{d}_i(x_i^j, x_i^k) + \sum_{i=M+1}^{\infty} 2^{-i}\hat{d}(x_i^j, x_i^k) \\ &\leq \sum_{i=0}^{M} d_i(x_i^j, x_i^k) + \sum_{i=M+1}^{\infty} 2^{-i} \\ &< M(\varepsilon/(2M)) + 2^{-M} \\ &< \varepsilon. \end{aligned}$$

Hence $((x_i^j))_{j\geq 1}$ is Cauchy.

Assume each (X_i, d_i) is complete. If $((x_i^j))_{j\geq 1}$ is a Cauchy sequence in $\prod_{i=1}^{\infty} X_i$ then, by above, each $(x_i^j)_{j\geq 1}$ is Cauchy in X_i. By assumption each $(x_i^j)_{j\geq 1}$ converges. By question 4.(b) it follows that $((x_i^j))_{j\geq 1}$ converges and so $\prod_{i=1}^{\infty} X_i$ is complete.

Now assume $(\prod_{i=1}^{\infty} X_i, d_\pi)$ is complete. For each $i \geq 1$ let $(x_i^j)_{j\geq 1}$ be a Cauchy sequence in X_i. Then the sequence $((x_i^j))_{j\geq 1}$ is Cauchy in $\prod_{i=1}^{\infty} X_i$. By assumption $((x_i^j))_{j\geq 1}$ is convergent. Then each $(x_i^j)_{j\geq 1}$ is convergent by problem 4.(b). Hence each X_i is complete.

6. Let $((x_i^j))_{j\geq 1}$ be a sequence in $\prod_{i=1}^{\infty} X_i$. We will contstruct, inductively, strictly increasing functions $\sigma_n : \mathbb{N} \to \mathbb{N}$ such that, for all $n \geq 1$, $((x_i^{\sigma_{n+1}(k)}))_{k\geq 1}$ is a subsequence of $((x_i^{\sigma_n(k)}))_{k\geq 1}$ and for each $i \geq 1$ the sequence $(x_i^{\sigma_i(k)})_{k\geq 1}$ converges in X_i.

Since X_1 is sequentially compact the sequence $(x_1^j)_{j\geq 1}$ has a convergent subsequence. Let $\sigma_1 : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $(x_1^{\sigma_1(k)})_{k\geq 1}$ is a convergent subsequence. Assume we have constructed $\sigma_1, \ldots, \sigma_n$ as above. Since X_{n+1} is sequentially compact $(x_{n+1}^{\sigma_n(k)})_{k\geq 1}$ has a convergent subsequence. Let $\sigma_{n+1} : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $(x_{n+1}^{\sigma_{n+1}(k)})_{k\geq 1}$ is a convergent subsequence of $(x_{n+1}^{\sigma_n(k)})_{k\geq 1}$.

For σ_n, $n \geq 1$, as above define the strictly increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ by $\sigma(k) = \sigma_k(k)$ and consider the subsequence $((x_i^{\sigma(k)}))_{k\geq 1}$ of $((x_i^j))_{j\geq 1}$. For any $i \geq 1$ and $n \geq i$ we have that $(x_i^{\sigma_n(k)})_{k\geq 1}$ is a subsequence of $(x_i^{\sigma_i(k)})_{k\geq 1}$. Hence the sequence $(x_i^{\sigma(k)})_{k\geq i}$ is a subsequence of $(x_i^{\sigma_i(k)})_{k\geq i}$. Since a subsequence of a convergent sequence converges to the same limit and convergence does not depend on any finite number of beginning terms, it follows that $(x_i^{\sigma(k)})_{k\geq 1}$ is convergent. By problem 4.(b), $((x_i^{\sigma(k)}))_{k\geq 1}$ is convergent. Hence $\prod_{i=1}^{\infty} X_i$ is sequentially compact.