Homework I: Mathematics 121

Due Friday, April 10, 2009

1. Suppose (X, d_X) and (Y, d_Y) are metric spaces. Let $X \times Y = \{(x, y) : x \in X, y \in Y\}$ (ordered pairs).

(a) Define $D : (X \times Y) \times (X \times Y) \to \mathbb{R}$ to be

$$D((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2).$$

Show $(X \times Y, D)$ is a metric space.

(b) Suppose $\{(x_j, y_j) : j = 1, 2, 3 \ldots\}$ is a sequence in $X \times Y$. Show that this sequence converges in $(X \times Y, D)$ to (x_0, y_0) if and only if: $\{x_j\}$ converges to x_0 and $\{y_j\}$ converges to y_0

2. Suppose every sequence in (X, d_X) has a convergent subsequent (i.e., a subsequence which converges to a point in X) and similarly for Y. Prove that every sequence in $(X \times Y, D)$ has a convergent subsequence.

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3. Suppose (X, d) is a metric space. Define

$$\hat{d} : X \times X \to \mathbb{R} \text{ by } \hat{d}(x,y) = \min(1, d(x,y))$$

Prove: $(X, \hat{d})$ is a metric space.

4. Suppose (X_i, d_i), $i = 1, 2, 3 \ldots$ is a sequence of metric spaces. Define the set ΠX_i to be

$$\{ (x_1, x_2, x_3 \ldots) : x_i \in X_i \text{ for all } i = 1, 2, 3 \ldots \}$$

[that is, an element of ΠX_i is a sequence with 1st element in X_1, 2nd element in X_2, etc].

(a) Define $d_\Pi : (\Pi X_i) \times (\Pi X_i) \to \mathbb{R}$ by

$$d_\Pi((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_1^{+\infty} \left(\frac{1}{2^i} \min(d_i(x_i, y_i), 1) \right).$$

Show that d_Π is defined (=the series converges) & that $(\Pi X_i, d_\Pi)$ is a metric space.

(b) Prove that a sequence of points in ΠX_i converges to a point in ΠX_i if and only if it converges "in each slot", namely, if the jth sequence (point in ΠX_i) is

$$(x_1^j, x_2^j, x_3^j, \ldots) \qquad x_i^j \in X_i, \forall i$$

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5. Prove that $(\prod X_i, d_\pi)$ is complete if and only if each (X_i, d_i) is complete.

*6. Suppose each (X_i, d_i) has the property that every sequence has a convergent subsequence. Prove that

$(\prod X_i, d_\pi)$ has the property that every sequence has a convergent subsequence.

[Suggestion: Find a (first) subsequence that converges in the first slot. Then find a second subsequence, namely a subsequence of the first subsequence, that also converges in the second slot. Continue — so the Nth subsequence converges in each of the first N slots. Then take the first element of the first subsequence, the second element of the second, the third element of the third subsequence, etc.]