General topological spaces: definitions and basic examples

Definition: A <u>topological</u> <u>space</u> is a set X together with a set $\mathcal{U}$ of subsets of X (to be called the "open sets in X") such that

(i) $\phi, X \in \mathcal{U}$

(ii) If $U_\lambda \in \mathcal{U}$, all $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{U}$.

(iii) If $U_1, U_2 \ldots, U_n \in \mathcal{U}$, then $\bigcap_{j=1}^{n} U_j \in \mathcal{U}$.

In words:

(i) ϕ and X are open

(ii) arbitrary union of open sets is open

(iii) finite intersection of open sets is open.

Example 0: X arbitrary set, $\mathcal{U} = \{\phi, X\}$ ("indiscrete topology").

Example 1: X a metric space, metric d, $\mathcal{U}$ = set of all d-open sets.
(includes: X arbitrary set, $(0,1)$ metric (all distances $= 0$ or 1), so $\mathcal{U}$ = all subsets of X (the discrete topology on X).

Note that if X contains more than one point, then the indiscrete topology is not a "metric topology" in the sense of Example 1.
Reason: p, q $p \neq q$ pts in $(X, d) \Rightarrow$ open $B(p, \frac{d(p,q)}{2}) \neq \phi$ (p is in it) and $\neq X$ (q is not in it).

Another kind of example: X a set, $U \neq \phi$, $U \subset X$ open if $X - U$ is "small" in some sense. If we take "small" to mean $= \phi$, we get example 0!

Other possibilities along this line:

Ex. 2(a) set X, $\mathcal{U} = \phi$ together with
2(a) all subsets with finite complement, e.g. $U \subset X$ is open if U is empty or $X \setminus U$ is finite. (only interesting if X is infinite)

Ex. 2(b) X set, $\mathcal{U} = \phi$ together with subsets U such that $X - U$ is empty, finite,
2(b) or countably infinite. (only interesting if X is uncountable: otherwise — if X is countable — this is the discrete topology).

Ex. 2(c) X a complete metric space, $\mathcal{U} = \phi$ together with sets U such that
2(c) $X \setminus U$ is "first category" for d, that is $X \setminus U$ = countable union of d-nowhere dense sets. (If X is not complete, this is potentially less interesting since it could be that every set in X was a countable union of nowhere dense sets, but if X is d-complete and especially if X is d-complete with no isolated points in the d metric, then this is more intriguing).

(Intuitive only):

2(d) If X is a metric space with dimension n, then $\mathcal{U} = \phi$ together with X and sets of the form $X - C$ where dimension $C < n$. This depends on some sensible idea of dimension (which we are not going to discuss)!

Definition: A set C in a topological space X is <u>closed</u> if and only if $X \setminus C$ is open.

Thus closed sets satisfy
(i) ϕ, X closed (ii) arbitrary intersection of closed sets is closed (iii) finite union of closed sets is closed.

We can define a topological space by choosing which sets are closed, with properties (i), (ii), (iii) here and then defining U to be open if $X \setminus U$ is closed.
Each of the (i), (ii) (iii) properties of closed sets gives the corresponding required properties for open sets, by de Morgan's Laws.

Examples 2(a)-2(d) can conveniently be thought of as specifying that closed sets are sets that are "small" in some sense.

Many of the concepts in metric space topology can be carried over directly to general topological spaces: as long as the concept is defined in terms of open sets in the metric space, it goes right over verbatim. For example:

Def: A set $C \subset X$, X a topological space, is <u>compact</u> if, for every collection $\mathcal{U}_\lambda, \lambda \in \Lambda$, of open sets in X such that

$$C \subset \bigcup_{\lambda \in \Lambda} \mathcal{U}_\lambda,$$

there is a finite set $\lambda_1, \dots, \lambda_k$ such that

$$C \subset \bigcup_{j=1}^{k} \mathcal{U}_{\lambda_j}.$$

(This can in particular be applied to $C = X$ to define when a topological space is compact).

Definition: $f: X \to Y$, X, Y topological spaces is <u>continuous</u> if: V open in $Y \Rightarrow f^{-1}(V)$ open in X, i.e. $f^{-1}(V)$ is open in X for each and every V open in Y.

Definition: X is <u>disconnected</u> if $\exists\, \mathcal{U}, V$ open in $X \ni \mathcal{U} \neq \phi$, $V \neq \phi$, $\mathcal{U} \cap V = \phi$ and $\mathcal{U} \cup V = X$.
X is <u>connected</u> if it is not disconnected.

Some things work the same way for topological spaces as for metric spaces. For instance,

Theorem: $f : X \to Y$, f continuous and onto, X compact $\Rightarrow$ Y compact.

Theorem: $f : X \to Y$, f continuous and onto, X connected $\Rightarrow$ Y connected.

Proofs: Exercises, following the metric space arguments (in their open set form).

But some things do not work when transferred to the general topological space setting.

Example: Let X be an infinite set with the "finite complement" topology of Example 2(a) above. Then every subset C of X is compact.

Proof: Suppose $\bigcup_{\lambda \in \Lambda} U_\lambda \supset C$. Choose $\lambda_0 \in \Lambda$.

Then $C \smallsetminus (C \cap U_{\lambda_0}) = C \cap (X \smallsetminus U_{\lambda_0})$ is finite, say, $= \{p_1, \ldots, p_\ell\}$. Choose $U_{\lambda_i} \ni p_i \in U_{\lambda_i}$, $i = 1, \ldots \ell$. Then

$$C \subset \bigcup_{j=0}^{\ell} U_{\lambda_j} \quad . \quad \square$$

But no infinite proper subset of X is closed.

So there are compact subsets of X which are not closed. (Contrast this with the fact that a compact subset of a metric space is always closed).

In some sense what is "wrong" with the infinite set, finite complement topology is that there are "not enough" open sets. In particular, the following property fails:

Definition: A topological space X is Hausdorff or "T_2" if for each $p, q \in X$, $p \neq q$ there are open sets U_p, U_q with $p \in U_p$, $q \in U_q$ and $U_p \cap U_q = \phi$.

Metric spaces are always Hausdorff; in (X, d) one can take $U_p = B(p, d(p,q)/2)$ $U_q = B(q, d(p,q)/2)$.

Theorem: If X is a Hausdorff topological space and C is a compact subset of X, then C is a closed set in X.

Proof: Let $y \in X - C$. For each $x \in C$, $x \neq y$, so $\exists$ U_x and V_x such that $x \in U_x$, $y \in V_x$, U_x & V_x open, and $U_x \cap V_x = \phi$. Clearly $\bigcup_{x \in C} U_x \supset C$. So $\exists\, x_1, \ldots, x_\ell$ $\ni C \subset \bigcup_{j=1}^{\ell} U_{x_j}$. Then $y \in \bigcap_{j=1}^{\ell} V_{x_j}$ and $\bigcap_{j=1}^{\ell} V_{x_j}$ is open.

Also $\left(\bigcap_{j=1}^{\ell} V_{x_j}\right) \cap \left(\bigcup_{j=1}^{\ell} U_{x_j}\right) = \phi$. Hence $\left(\bigcap_{j=1}^{\ell} V_{x_j}\right) \cap C = \phi$.

Let $V_y' = \bigcap_{j=1}^{\ell} V_{x_j}$ as constructed for each $y \in X \sim C$. (The value of ℓ will depend on y, etc.). Clearly $V_y \subset X - C$. But $y \in V_y$.

So $\bigcup_{y \in X \sim C} V_y = X \sim C$. Thus $X \sim C$ is a union of open sets and hence is open. So C is closed. $\square$

(This proof is also in the book!)