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# Outline of Linear Algebra: Part I

**Prologue:** Determinants: definition, multilinearity, antisymmetry, row = column

Gaussian idea. Application: Determinant = 0 $\iff$ rows dependent

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**Definition & examples** Vector space over field.

Fundamental examples:

$\mathbb{R}^n$, E over F, $l_2$, $C([0,1])$, free vector space, (includes eventually zero sequences), solutions of linear ordinary & partial differential equations

$l_2$ over $\mathbb{C}$, complex-valued function spaces

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**Basis & dimension** Fundamental fact: If $v_1, \dots, v_k$ generate $V$ and if $n > k$, then $w_1, \dots, w_n$ is a linearly dependent set.

Proof: More unknowns than eq $\Rightarrow$ nontriv solution of homogeneous system <u>or</u> replacement proof.

Consequences: If $V$ is generated by a finite set, then $\exists$ (unique) number $n$ such that

a. $\exists$ a linearly independent set $v_1, \dots, v_n$ generating $V$.

b. No smaller set can generate $V$

c. No larger set can be linearly independent

d. Any generating set with $n$ vectors is independent

e. Any independent set with $n$ vectors generates.

$n$ = dimension of $V$ (by definition).

$V$ of dimension $n$ is "isomorphic" to $\mathbb{R}^n$ (or $F^n$ if $F$ = field)

Namely if $v_1, \ldots, v_n$ is a basis (generating, linearly independent set), then each $v \in V = \sum \alpha_i v_i$
$\alpha_i$ uniquely determined and

$$v \longrightarrow (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$$

is bijective & linear (with linear inverse): v.space operations are preserved!

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Linear transformations

General idea of linear transformation $T : V \to W$.
$T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$.
$\ker T \underset{\text{def}}{=} \{v : Tv = 0\}$.
range$T$ (= image$T$ = im$T$) $\overset{\text{def}}{=} \{Tv : Tv \in W\}$.
$\ker T$ & range $T$ are subspaces (subvector spaces)
Basic observation: If $V$ is finite dimensional, then so are $\ker T$ & range$T$ and

$$\dim(V) = \dim(\ker T) + \dim(\text{range} T).$$

Proof: Choose $v_1, \ldots, v_k$ basis for $\ker T$, extend to $v_1, \ldots, v_k, v_{k+1} \ldots v_n$ basis for $V$, then

$T|\,\text{span}(v_{k+1}, \ldots, v_n)$ is 1-1 onto range$(T)$ so
$\dim(\text{range} T) = n - k$. $\square$

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Matrix of a linear transformation

Matrix of a linear transformation $T : V \to W$
basis $v_1, \ldots, v_n$ for $V$, $w_1, \ldots, w_m$ for $W$:
$A_{ij}$ determined by

$$T v_j = \sum_{i=1}^{m} A_{ij} w_i \qquad j = 1, \ldots n$$

(No. of columns = $\dim V$, no. of rows = $\dim W$
columns of matrix are images of $v$'s in $w$-component form).

Note how this works: a vector $v = \sum \alpha_i v_i$ has $T$-image in $W$ coordinates, thought of as a "column vector" obtained by

$$\alpha_1 (\text{1st column of } A) + \alpha_2 (\text{second column of } A) + \dots + \alpha_n (\text{last column of } A)$$

This corresponds to usual "across first, down second factor) matrix product

$$\underset{\text{row}\nearrow\ \uparrow\text{column}}{\left(A_{ij}\right)} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \text{column representing } Tv \text{ in } W\text{-components.}$$

Example: $T: \mathbb{R}^3 \to \mathbb{R}^2$ with $T((1,0,0)) = (2,3)$ $T((0,1,0)) = (-1,+1)$ and $T((0,0,1)) = (-5,7)$ has matrix

$$\begin{pmatrix} 2 & -1 & -5 \\ 3 & +1 & 7 \end{pmatrix}$$

so that indeed

$$\begin{pmatrix} 2 & -1 & -5 \\ 3 & +1 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{etc.}$$

Note: Other conventions would be possible (e.g., multiply on right by matrix etc.)

Obvious: Every linear transformation is given by a matrix once bases are chosen, and every matrix is the matrix of some linear transformation, this being unique (again with bases chosen).

Note: The matrix idea is really only useful in finite-dimensional situation

Composition of linear transformations:

$$V_1 \xrightarrow{T} V_2 \xrightarrow{S} V_3 \qquad S \circ T \quad \text{(notation)}$$

matrix of $S \circ T$ is $BA$ where $A$ = matrix of $T$ and $B$ = matrix of $S$. (matrix on right acts first! $(BA)\vec{v} = B(A\vec{v})$, $\vec{v}$ = column vector! (matrix multiplication is associative). Note order is reversed in a sense. And order matters: matrix multiplication is not commutative (and linear transformations are not) even when everything is defined (and it might not be!).

Example: $T((x,y)) = (0,y) \qquad S((x,y)) = (y,x)$

Then $S(T((x,y)) = S((0,y)) = (y,0)$ while $T(S((x,y))) = T((y,x)) = (0,x)$. These are equal only at $(0,0)$!

Note: "Generic" matrices fail to commute.

Dual space: $V$ a vector space, $V^* \stackrel{\text{def}}{=}$ the set of linear transformations $T$ of $V$ to $\mathbb{R}$ (or whatever field $V$ is defined over). $V^*$ is a vector space with the obvious ideas of addition and multiplication by scalars.

Note: A fixed but arbitrary $v_0 \in V$ determines (defines) a linear transformation $T_{v_0}: V^* \to \mathbb{R}$ by $T_{v_0}(\omega) = \omega(v_0)$, $\omega \in V^*$

Thus $V$ "$\subset$" $(V^*)^*$ in some natural sense of $\subset$.

If $V$ is infinite dimensional, then $(V^*)^*$ is larger than $V$. But if $V$ is finite dimensional $V$ "$=$" $(V^*)^*$

as we shall see momentarily (where "=" is in the same sense as "⊂": $V$ is naturally identified with $(V^*)^*$ by sending each $v_0$ to $T_{v_0}: V^* \to \mathbb{R}$ (or $F$)).

Definition: If $V$ is finite dimensional and $v_1, \ldots, v_n$ is a basis, then the <u>dual basis</u> of $V^*$ is $\omega_1, \ldots, \omega_n$ where $\omega_j$ is defined by $\omega_j(v_j) = 1$, $\omega_j(v_k) = 0$ if $k \neq j$. [Note: This determines $\omega_j$ since $\omega(\Sigma \alpha_i v_i) = \Sigma \alpha_i \, \omega(v_i)$ for each $\omega \in V^*$; and the $\omega(v_i)$ can be chosen arbitrarily to use this formula as a definition of an $\omega \in V^*$]. Easy to check: $\omega_1, \ldots, \omega_n$ really is a basis for $V^*$.

So $\dim V^* = \dim V$.

The spaces $V^*$ & $V$ are isomorphic but not "canonically isomorphic" since the isomorphism depends on basis choice. But if $V$ is finite dimensional, then $V$ and $V^{**}$ are canonically isomorphic, since $v_0 \to T_{v_0}$ (as earlier) does <u>not</u> depend on basis choice.

Adjoints: If $T: V \to W$ is a linear transformation, then the <u>adjoint</u> of $T$ is the linear transformation $T^*: W^* \to V^*$ defined by $T^*(\omega)(v) = \omega(T(v))$, $\omega \in W^*$, $v \in V$. Note that $T^*(\omega) \in V^*$ as required.

Slightly tedious but routine exercise: If $T: V \to W$ ($V, W$ finite dimensional) has matrix $A$ relative to bases $v_1, \ldots, v_n$ of $V$ and $w_1, \ldots, w_n$ of $W$, then $T^*$ has matrix $A^t$ ($A$ transpose) relative to the associated dual bases of $V^*$ and $W^*$, where $(A^t)_{ij} = A_{ji}$ (note number of rows of $A^t$ = number of columns of $A$, number of columns of $A^t$ = number of rows of $A$). Clearly $(A^t)^t = A$. Correspondingly $(T^*)^* = T$ with $(V^*)^*$ and $(W^*)^*$ being identified with $V$ and $W$ canonically. Note this is for $V$, $W$ finite dimensional only! (the main case we are interested in).

Null space of adjoint etc. (For most of the following $V, W$ finite dimensional until further notice!)

Basic definition: If $S$ is a subset of $V$, the annihilator of $S$ is a vector subspace of $V^*$ defined as $\{T \in V^* : T(v) = 0 \text{ for all } v \in S\}$. Notation: $S^\perp$. Note that $S^\perp = V_1^\perp$ where $V_1 = \text{span } S$ = smallest vector subspace of $V$ containing $S$. So one really might as well just consider annihilators of subspace.

Basic fact :: $\ker T^* = (\text{image } T)^\perp$.

Proof: If $\omega \in (\text{image } T)^\perp$, then for each $v \in V$, $\underbrace{(T^*\omega)(v) =}_{\text{by definition}} \omega(T(v)) = 0$, so $\omega \in \ker T^*$. Conversely, if $\omega \in \ker T^*$ then $0 = (T^*\omega)(v) = \omega(T(v))$ for all $v \in V$. So $\omega \in (\text{image } T)^\perp$.

[Up to here, finite dimensionality has not been used!" But now it will be].

Easy fact (from before) $\dim \ker T + \dim \text{image}\, T$
$= $ dimension $V$.
Second easy fact: $V_1 \subset V$, $\dim V_1^\perp =$
$n - \dim V_1$, $n = \dim V$. (Proof: Choose basis of $V_1$, extend to basis of $V$, look at dual basis $\omega_1, \dots, \omega_k, \omega_{k+1} \dots \omega_n$, $k = \dim V_1$. Clear that $\omega_{k+1}, \dots, \omega_n$ are basis for $V_1^\perp$.)
$n = \dim V$, $m = \dim W$ ($\Rightarrow \dim V^* = n$, $\dim W^* = m$)
Corollary: $\dim \ker T^* = \quad . \quad \dim (\text{image}\, T)^\perp$
$= \quad (m - \dim \text{image}\, T)$
So $\dim \text{image}\, T^* = m - \dim \ker T^*$
$= m - (m - \dim(\text{image}\, T)) = \dim(\text{image}\, T)$
So image $T^*$ and image $T$ have the same dimension.
Translated to matrix terms, this is
"row rank = column rank":
max no of linearly independent columns $= \dim \text{image}\, T$
max no of linearly independent rows $=$
max no of linearly independent columns of $A^t$
$= \dim \text{image}\, T^*$.
This common value is called the rank of the matrix A.