Sample Problems for Midterm I

1. Find all functions that are
   (1) equal to their own first derivative
   (2) equal to their own second derivative
   (3) equal to their own third derivative.

2. Discuss the solution(s) of the equation
$$\frac{d^2y}{dx^2} + k^2 y = \sin bx$$
where k, b are positive constants.

3. Show that every solution of
$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$
$a, b > 0$, is "transient", i.e., has limit $= 0$ as $x \to +\infty$.

4. State the basic existence and uniqueness result of ordinary differential equations.

5. Using the uniqueness part of 4, explain why
$$\det\begin{pmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & \cdots & & & y_n' \\ y_1^{(n-1)} & \cdots & & & y_n^{(n-1)} \end{pmatrix}$$
is either everywhere nonzero or everywhere 0 where $y_1, \ldots, y_n$ each solve $\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_n y = 0$.

[Here $y_n^{(n-1)} = \frac{d^{(n-1)}y}{dx^{n-1}}$ etc.] Suggestion for this: if $\det = 0$ at one point, then some linear combination of y_j's $\equiv 0$.

(Suggestion for 5 continued): This comes from the fact that $\det = 0$ at $x = x_0$ implies that for some $c_1, \ldots, c_n$ not all 0

$$\sum_{j=1}^{n} c_j \begin{pmatrix} y_j \\ y_j' \\ \vdots \\ y_j^{(n-1)} \end{pmatrix} \Bigg|_{x_0} = 0 \quad \Big(\text{or} \quad \sum c_j y_j \big|_{x_0} = 0 \quad \sum c_j y_j' \big|_{x_0} = 0 \quad \cdots \quad \sum c_j y_j^{(n-1)} \big|_{x_0} = 0 \Big)$$

6. Suppose $y_1 \ldots y_n$ are solutions of $\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0$ with $y_1(x_0) = 0$, $y_2(x_0) = 0$, $\ldots$ $y_n(x_0) = 0$. Show that $y_1, \ldots, y_n$ are dependent, i.e. there are constants $c_1 \ldots c_n$ not all 0 such that $\sum_{j=1}^{n} c_j y_j \equiv 0$.

7. Show by induction that $(D - \alpha)^k (x^k e^{\alpha x}) = k! e^{\alpha x}$.

8. Use problem 7 to show that if $P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$ and $P(\lambda) = (\lambda - \alpha)^k Q(\lambda)$ where $Q(\alpha) \neq 0$ then $y = \frac{1}{k!\, Q(\alpha)} x^k e^{\alpha x}$

solves
$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$$

(here a's are constants).

9. Show that your solution of problem 2 is a special case of problem 8.

10. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

by variation of parameters. Check your answer!

11. Explain why $\det\begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & & \lambda_n^{n-1} \end{pmatrix} \neq 0$ if the $\lambda_1, \ldots, \lambda_n$ are all distinct.

In terms of algebra (as done in class).

12. Prove that if $\lambda_1, \ldots, \lambda_n$ are distinct real numbers then $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ are linearly independent by using "order of magnitude" reasoning (cf. the handout and class notes — done in class)

13. Observing that $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ all solve

$$\frac{d^n y}{dx^{n-1}} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = 0$$

where $P(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$ has coefficients $1, a_1, \ldots, a_n$ $(P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n)$

discuss how problems 12, 11, and 5 are related.

14. Consider the equation, with $\tau, k > 0$,

$$\frac{d^2y}{dx^2} + \tau\frac{dy}{dx} + ky = \sin(\omega t)$$

(a) Show that the solution (general solution)

14 (continued) consists of a transient part + a "sinusoidal" function, of the form $C \sin(\omega t - \phi)$.

(b) Find the ω (in terms of τ, k) for which the nontransient part has a maximum amplitude under the conditions when such a maximum amplitude value of ω exists. (Problem includes finding the conditions: cf. prob 1, p. 113).

15. Prove that $\int_{-\pi}^{+\pi} \sin kx \cos lx \, dx = 0$ k, l integers > 0

also

$$\int_{-\pi}^{+\pi} \sin kx \sin lx \, dx = 0 \quad \text{if } k \neq l$$

$$\int_{-\pi}^{+\pi} \cos kx \cos lx \, dx = 0 \quad \text{if } k \neq l.$$

16. Explain why

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) \, dx$$

$$a_j = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos jx \, dx$$

$$b_j = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin jx \, dx$$

are a good guess for trying to write

$$f(x) = a_0 + \sum_{j=1}^{+\infty} a_j \cos(jx) + \sum_{j=1}^{+\infty} b_j \sin(jx).$$

17. Let $f(x) = a_0 + \sum_1^N a_j \cos(jx) + \sum_1^N b_j \sin(jx)$ (a's & b's real constants). Show that

$$\int_{-\pi}^{+\pi} f^2(x) \, dx = 2\pi a_0^2 + \pi \left(\sum_1^N a_j^2 + \sum_1^N b_j^2 \right).$$

18. Find the Fourier series of the function
$f(x) = x$ on $[-\pi, +\pi]$

19. Find the Fourier series of the function
$f(x) = +1$ if $x \in (0, +\pi]$
$f(x) = -1$ if $x \in (-\pi, 0]$.

20. Find the Fourier series of $f(x) = |x|$ on $[-\pi, +\pi]$

21 (a) Show that variation of parameters applied to $\frac{d^2y}{dx^2} + y = f(x)$
gives the particular solution
$$y(x) = \int_0^x f(t) \sin(x-t)\, dt$$

Note: x occurs twice here as upper limit & inside.

(b) Check by calculation that this solution actually works (this involves "differentiating under the integral sign" as well as relative to the upper limit).