Riemannian Metrics and Differential Forms

If V is a (finite dimensional) vector space with inner product $\langle\, ,\rangle$, we can put an inner product structure on $\Lambda^k V^*$ (= antisymmetric multilinear maps $V \times \ldots \times V \to \mathbb{R}$ (k copies)) as follows: Choose $v_1, \ldots v_n \in V$ orthonormal basis for V, $\theta_1, \ldots \theta_n$ its dual basis and *define* $\theta_{i_1} \wedge \ldots \wedge \theta_{i_k}$, $i_1 < \ldots < i_k$, to be orthonormal for $\Lambda^k V^*$ — defining an orthonormal basis, or more precisely defining a given basis to be orthonormal, define an inner product. But of course this is interesting only if the resulting inner product on $\Lambda^k V^*$ does not depend on the choice of orthonormal basis $v_1, \ldots v_n$ for V itself. It actually does not depend on the choice of basis for V, but this requires a bit of maneuvering to prove. We can think of this question as follows: Another (any other!) orthonormal basis of V looks like $(\sum_j A_1^j v_j, \sum_j A_2^j v_j, \ldots, \sum_j A_n^j v_j)$ where the matrix A_i^j is orthogonal. Such a matrix induces a transformation on V^* which is also orthogonal (easy exercise), that is, has an orthogonal matrix as its matrix representation. This in turn induces a transformation on $\Lambda^k V^*$, and we want to see that the matrix of that transformation is orthogonal.

This looks like a mess. It amounts to something like saying that the $k \times k$ signed minors of an orthogonal $n \times n$ matrix (i.e. signed $k \times k$ submatrix determinants) themselves form an orthogonal $\binom{n}{k} \times \binom{n}{k}$ matrix.

But we can use what we know about the orthogonal group to simplify this. First of all, by real analyticity, it is enough to check this for all orthogonal transformation in a neighborhood of the identity (this gives $SO(n)$ — but the conclusion is obvious for $\begin{pmatrix} -1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & 0 & \ddots & 1 \end{pmatrix}$ (one sign change of v, only) so if we are all right for $SO(n)$ we are also all right for $O(n)$). So it is enough to check 1-parameter subgroups of $SO(n)$ (near the identity) since $\bigcup$ 1 parameter subgroups $\supset$ neighborhood of the identity. Actually $\bigcup$(1-parameter subgroups) $\supset$ all of $SO(n)$, but we have not proved this (it follows from canonical form things, but in any case we do not need it). To check the conclusion for 1-parameter subgroups, we saw earlier that all we need is information about the infinitesimal generator of the 1-parameter subgroup. Specifically, suppose γ_t is a 1-parameter subgroup of $SO(n)$.

Then $\gamma(t) = \exp(tA)$ for some skew-symmetric $n \times n$ matrix A. Associated to γ is a one-parameter subgroup of $Gl(\binom{n}{k}, \mathbb{R})$ acting on $\Lambda^k V^*$, call it $\hat{\gamma}_A(t)$. We then know that $\hat{\gamma}_A(t) \in SO(\binom{n}{k})$ for all t if (and only if) $\frac{d}{dt}\hat{\gamma}_A(t)\big|_{t=0}$ is a skew-symmetric matrix in $Gl(\binom{n}{k}, \mathbb{R})$.

Now the association $A \to \frac{d}{dt}\hat{\gamma}_A(t)\big|_{t=0}$ is a linear map: it is a differential of a smooth map. So to check that $\frac{d}{dt}\hat{\gamma}_A(t)\big|_{t=0}$ is skew-symmetric for all skew-symmetric A we need only check that this is true for A varying over a basis for the set of skew-symmetric $n \times n$ matrices. There is a natural set of matrices to use for a basis, namely $A(i,j)$, $i \neq j$, $i < j$, with the ij entry of $A(i,j) = +1$ and the ji entry $= -1$. The associated ~~rotation~~ one parameter subgroup γ_{ij} in $SO(n)$ is rotation in the i,j variables, no action in other variables. But $\hat{\gamma}_{ij}$ acts on $\Lambda^k V^*$ orthogonally by an easy inspection! (one is really looking at a two variable situation: elements with θ_i only or θ_j only are acted on by "rotation" in the θ_i, θ_j plane, $\theta_i \wedge \theta_j \wedge$ (something else) are fixed & if no $\theta_i \wedge \theta_j$ or is in a term it is fixed). Hence $\frac{d\hat{\gamma}_{ij}(t)}{dt}\Big|_{t=0}$ is skew-symmetric. (You could also look at this directly, without going to $\hat{\gamma}_{ij}(t)$, but just looking directly at its derivative). So $\frac{d}{dt}\hat{\gamma}_A(t)\Big|_{t=0}$ is skew-symmetric for all skew-symmetric A. This looks like magic, but it works!

Taking this over to the manifold setting, we now have that if M is a Riemannian manifold, then there is an induced inner product on the k-forms at each $p \in M$. In particular, if M^n is orientable and oriented, then there is a C^∞ n-form Ω on M, nowhere vanishing on M, determined by $\|\Omega(p)\| = 1$ $\left(\| \ \| = \underset{\text{as usual}}{<\ \ >^{\frac{1}{2}}}\right)$ and $\Omega(p) > 0$ relative to the orientation, $\forall\, p \in M$. This form is called the (Riemannian) volume form. [If M is not oriented, nor orientable even, one still gets Ω up to $\pm$, so one has a measure locally and hence globally defined locally by, U orientable open set, $\mathrm{vol}(U) = \int_U \omega$ where $\|\omega\| = 1$ and $\omega > 0$ relative to some orientation of U, same one used for $\int_U$).

From now on, we assume M is an orientable Riemannian manifold with a fixed orientation.
We want to define an operation $*$ that operates pointwise to take k-forms to $n-k$ forms, $n = \dim M$.
Namely if $\theta_1, \dots \theta_n$ is an orthonormal basis, + oriented, for T_pM^*, we set $*(\theta_{i_1} \wedge \cdots \theta_{i_k}) = \pm$ wedge of θ's in order which are not among $\theta_{i_1}, \dots, \theta_{i_k}$
where $\pm$ is chosen so that

$$\theta_{i_1} \wedge \dots \wedge \theta_{i_k} \wedge *(\theta_{i_1} \wedge \cdots \wedge \theta_{i_k})$$
$$= \theta_1 \wedge \dots \wedge \theta_n \quad \text{(in order!)}$$

This looks as though it might depend on the choice of basis $\theta_1, \dots \theta_n$. But note that if α is a k-form (at p) and $*$ is defined relative to a particular basis (oriented) $\theta_1, \dots \theta_n$ then, for any $(n-k)$-form β

$$(+) \qquad \langle *\alpha, \beta \rangle \text{ volume form } = \alpha \wedge \beta$$

This is easy to check directly: it is enough to check $\alpha = \theta_{i_1} \wedge \dots \wedge \theta_{i_k}$, $\beta = \theta_{j_1} \wedge \dots \wedge \theta_{j_{n-k}}$. $\alpha \wedge \beta$ is nonzero only if i's $\cup$ j's $= 1, \dots n$, as is $\langle *\alpha, \beta \rangle$.
In the case i's $\cup$ j's $= 1, \dots n$, $*\alpha = \pm$ θ_j's wedge in order

where $\pm$ is determined by $\theta_{i_1} \wedge \cdot \theta_{i_k} \wedge$ (θ_j's in order) $= \pm$ volume. But this is also what make RHS $= \pm$ volume form.
The equation $(+)$ characterizes $*\alpha$ since it gives the inner product of $*\alpha$ with every $(n-k)$ form β.
And $(+)$ does not depend on basis choice!

Exercise: $\langle \alpha, \beta \rangle$ volume form $= \alpha \wedge *\beta$
(same kind of argument)

It is easy now to see that $*(*\alpha) = \pm\alpha$ where $\pm$ depends only on n and $\deg\alpha = k$, α a k-form. (complement of complement gives original set back!)

Now suppose α is a k-form, β a $(k+1)$ form

Then $\alpha \wedge *\beta$ is an $n-1$ form: $\deg\alpha = k$, $\deg *\beta = n-(k+1)$. So <u>if</u> <u>M</u> <u>is</u> <u>compact</u>

$$\int_M d(\alpha \wedge *\beta) = 0$$

But $d(\alpha \wedge *\beta) = d\alpha \wedge *\beta \pm \alpha \wedge d*\beta$

Hence $\int d\alpha \wedge *\beta = \mp \int \alpha \wedge d*\beta$

or $\int_M \langle d\alpha, \beta\rangle \Omega = \pm \int \langle \alpha, *d*\beta\rangle \Omega$, $\Omega =$ volume form

where $\pm$ depends only on degree of forms (on k & n)

Here we are using $d\alpha \wedge *\beta = \langle d\alpha, \beta\rangle$ vol. form

etc. (Exercise! from previous page)

So if we define the integrated inner product

$$\langle\langle \theta_1, \theta_2 \rangle\rangle = \int_M \langle \theta_1, \theta_2 \rangle \text{ volume form}$$

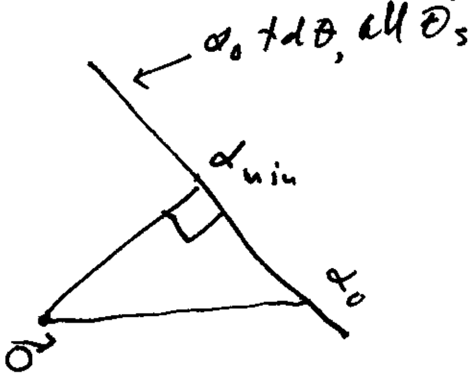
θ_1, θ_2 same degree,

we get that $\pm *d*$ is adjoint for d relative to this integrated inner product.

Now we look at the question of minimizing $\langle\langle \alpha, \alpha \rangle\rangle$ as α ranges over a deRham cohomology class, which has the form $\alpha_0 + d\theta$.

From here on, M is compact !!

By usual geometry, the minimum α_{min} if any has to be $\perp$ to range of d. By usual linear algebra, $(\text{range of } d)^\perp$



$=$ kernel of adjoint of d

$=$ (as earlier) kernel of d^*

So $\alpha_{min} \in \ker d \cap \ker(d^*)$. Of course, this is really only an expectation by analogy with finite dimensions since things like ker adj of $d = (\text{range of } d)^\perp$ are not really necessarily right in infinite dimensions.

But actually this all works:

Theorem (Hodge): In each deRham cohomology class, there exists a unique element (k form) α_{min}, C^∞, with $d\alpha_{min} = 0$ and $d^* \alpha_{min} = 0$.

(Forms satisfying $d = 0$ and $d^* = 0$ are called <u>harmonic</u> forms).

Actually, uniqueness is not hard to prove. (It is existence that makes this a famous theorem).

Uniqueness proof: If $\alpha_1 - \alpha_2 = d\theta$, α_1, α_2 harmonic

then $\langle\langle \alpha_1 - \alpha_2, d\theta \rangle\rangle = \|\alpha_1 - \alpha_2\|^2$

and $\langle\langle \alpha_1 - \alpha_2, d\theta \rangle\rangle = \langle\langle (\text{adj of } d)(\alpha_1 - \alpha_2), \theta \rangle\rangle$

$= 0$ since harmonic $\Rightarrow$ (adj of d)$(\alpha_1 - \alpha_2)$

$= \pm * d * \alpha_1 \mp * d * \alpha_2 = 0$ since $d * \alpha_1 = 0$ & $d * \alpha_2 = 0$.

Note that $*$ harmonic form is harmonic:

$d\alpha = 0 \Rightarrow d * (*\alpha) = 0$ since $** = \pm 1$

$(d*)\alpha = 0 \Rightarrow d(*\alpha) = 0$

So $*$ maps harmonic k-forms to harmonic $n-k$ forms

This is 1-1 onto since $** = \pm 1$ again.

So $$H^k_{\text{deRham}}(M, \mathbb{R}) = H^{n-k}_{\text{deRham}}(M, \mathbb{R})$$

when M is compact, oriented.

This is "Poincare duality", a famous result in manifold topology (the theorem as such actually pre-dates harmonic forms & deRham cohomology and was originally about topological cohomology).