Pullbacks, d, and induced maps on deRham Cohomology

Recall: If $F: M_1 \to M_2$ is a C^∞ map and ω is a k-form on M_2, then $F^*\omega$ is a k-form on M_1 defined by

$$F^*\omega|_p(v) = \omega|_{F(p)}(dF(v)) \qquad p \in M, v \in T_pM$$

(or $\omega|_{F(p)} F_* v$)

F^* behaves well in relation to d:

Lemma: $d_{M_1}(F^*\omega) = F^*(d_{M_2}\omega)$.

Slogan: "d commutes with pullbacks"

Proof: F^* obviously commutes with $\wedge$ products. Using (repeatedly) that $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2$, we conclude that it is enough to check that $d_{M_1}(f \circ F) = F^*(d_{M_2} f)$. To check this, note that if $p \in M_1$, $v \in T_pM_1$ then

$f: M_2 \to \mathbb{R}$

$$d_{M_1}(f \circ F)\Big|_p(v) = v(f \circ F)\Big|_p \overset{\text{definition of } F_*!}{=} (F_* v)f \underset{\text{def of } F^*}{=} F^*(d_{M_2} f)\Big|_p(v)$$

Recall definitions:

A form is <u>closed</u> if $d(\text{form}) \equiv 0$.

A form is <u>exact</u> if form = d(form of one lower degree)

and $H^k_{\text{deRham}}(M, \mathbb{R}) \overset{\text{def.}}{=} \dfrac{\text{closed } k \text{ forms on } M}{\text{exact } k \text{ forms on } M}$

The fact that d and F^* commute leads immediately to: (with $F: M_1 \to M_2$ as before)

(1) F^* (closed form on M_2) is a closed form on M_1

(2) F^* (exact form on M_2) is an exact form on M_1.

Hence F^* induces a map (also denoted by F^*) from $H^k_{deR}(M_2, \mathbb{R})$ to $H^k_{deR}(M_1, \mathbb{R})$.

This behaves well with respect to composition: If $F: M_1 \to M_2$ and $G: M_2 \to M_3$ then
$F^* \circ G^* = (G \circ F)^*$ both as a map on forms and also as maps on deRham cohomology.

Corollary: If $F: M_1 \to M_2$ is a diffeomorphism, then F^*: k-deRham for M_2 $\to$ k-deRham for M_1 is an isomorphism for each k.

Curiously, this conclusion holds if F is just a homeomorphism: deRham cohomology is a topological invariant, not just a diffeomorphism invariant.

N.B.: The distinction is not vacuous. $\exists$ C^∞ manifolds which are homeomorphic but not diffeomorphic.

Suppose ω_1 is a closed k-form and ω_2 a closed l-form. Then $\omega_1 \wedge \omega_2$ is a closed $(k+l)$ form since
$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2 = 0$
if $d\omega_1 = 0$ and $d\omega_2 = 0$.

It is not hard to see that if we write $[\omega]$ for the element-arising-from-a-closed-form ω in $H^k_{deR}(M,\mathbb{R})$, then $\wedge$ induces a product operation of $H^k(M,\mathbb{R}) \times H^l(M,\mathbb{R})$.

Namely $[\omega_1] \wedge [\omega_2] = [\omega_1 \wedge \omega_2]$.

(One needs to check this is "well defined". Namely if ω_1 is replaced by $\omega_1 + d\theta_1$ and ω_2 by $\omega_2 + d\theta_2$ then $(\omega_1 + d\theta_1) \wedge (\omega_2 + d\theta_2) - \omega_1 \wedge \omega_2$ is exact.

But $(\omega_1 + d\theta_1) \wedge (\omega_2 + d\theta_2) - \omega_1 \wedge \omega_2$

$$= \omega_1 \wedge d\theta_2 + d\theta_1 \wedge \omega_2 + d\theta_1 \wedge d\theta_2$$

and

$$d\theta_1 \wedge d\theta_2 = d(\theta_1 \wedge d\theta_2)$$
$$d\theta_1 \wedge \omega_2 = d(\theta_1 \wedge \omega_2)$$
$$\omega_1 \wedge d\theta_2 = (-1)^{\deg \omega_1} d(\omega_1 \wedge \theta_2)$$

from repeated application of the Leibnitz formula.

Thus one gets a ring (or algebra) structure on deRham cohomology as a whole, that is

$$\bigoplus_{k=0}^{n} H^k_{deR}(M,\mathbb{R}), \quad n = \dim M.$$

(k-forms above degree n are automatically 0, so the direct sum is finite). And F^* induces an algebra homomorphism from the deRham cohomology algebra for M_2 to that for M_1 (when $F: M_1 \to M_2$ is a C^∞ mapping).